

RIGIDITY OF AN ISOMETRIC $\mathrm{SL}(3, \mathbb{R})$ -ACTION

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ABSTRACT. We characterize the universal covering of connected analytic pseudo-Riemannian manifolds which admit a non-trivial and isometric action of the simple Lie group $\mathrm{SL}(3, \mathbb{R})$ with a dense orbit preserving a finite volume. If such manifold is also weakly irreducible we prove that M is isometric to, or a quotient space of, a simple Lie group containing $\mathrm{SL}(3, \mathbb{R})$.

INTRODUCTION

Let G be a connected non-compact simple Lie group acting on a connected, analytic manifold M preserving a pseudo-Riemannian structure. It is conjectured that such actions are rigid in the sense that they restrict the possibilities of the manifold. It is expected that any such actions, under non-trivial conditions, must be an algebraic double coset. That is, $M \cong K \backslash H / \Gamma$, such that H is a Lie group and there is a group homomorphism $G \hookrightarrow H$ whose image commutes with K , a compact subgroup of H , and $\Gamma \subset H$ is a lattice. Therefore, the G -action is given by a natural left action on $M \cong K \backslash H / \Gamma$. Some results in this direction are obtained in [7] and [8], proving that with some extra geometric conditions such G -actions imply that the manifolds are of double-coset type.

In [7], for M a complete and weakly irreducible manifold with a non-transitive G -action and with a dense orbit, it is proved that the dimension of M has a lower bound, in terms of the theoretical properties of the corresponding representation of \mathfrak{g} , the Lie algebra of G . In particular, [7] explains in detail the case $G = \widetilde{SO}_0(p, q)$, with $p + q \geq 4$, determining the manifold M .

The paper applies the techniques from [7] to the case $G = \mathrm{SL}(3, \mathbb{R})$, a connected, simple Lie group with Lie algebra $\mathfrak{sl}(3, \mathbb{R})$. In our case, $\mathrm{SL}(3, \mathbb{R})$ acts isometrically on M , a complete, pseudo-Riemannian manifold such that $8 < \dim(M) \leq 14$ with a dense orbit. The assumption on the dimension of M eliminates the condition of non-transitivity of the action.

In [7] and [8] we observe the study of irreducible representations of groups preserving a non-degenerate, symmetric, bilinear form. Such study is an important tool to understand the normal bundle of the foliation generated by the action of the group. That understanding together with the properties of the tangent bundle of the foliation give us information which restrict the possibilities of the manifold M .

Here, we analyze the representation of $\mathrm{SL}(3, \mathbb{R})$ of minimal dimension with the property of preserving a non-degenerate, symmetric bilinear form, which is non-irreducible. Hence, this paper generalizes such corresponding part of the previous works.

As in [7], we obtain a lower bound on the dimension of M which strongly determines \widetilde{M} , the universal covering of M . This is the result of our main theorem.

Theorem A. *Let M be a connected analytic pseudo-Riemannian manifold. Suppose that M is complete, has finite volume and admits an analytic and isometric $SL(3, \mathbb{R})$ -action with a dense orbit. If $8 < \dim(M) \leq 14$, then \widetilde{M} is isomorphic to one, and only one, of the following:*

- (i) $\widetilde{SL}(3, \mathbb{R}) \times \widetilde{N}$, where \widetilde{N} is a complete pseudo-Riemannian manifold.
- (ii) $G_{2(2)}$, the simply connected real Lie group related to the non-compact real form of the exceptional simple Lie algebra $\mathfrak{g}_2^{\mathbb{C}}$.
- (iii) $\mathbb{R} \backslash \widetilde{SL}(4, \mathbb{R})$.

Recall that a connected pseudo-Riemannian manifold is *weakly irreducible* if there is no proper, non-degenerate invariant subspace of the tangent space at some (at hence any) point invariant under the restricted holonomy group at that point. If in the hypothesis of Theorem A we assume that the manifold M is weakly irreducible then case (i) is not allowed, that is:

Theorem B. *Let M be a connected analytic pseudo-Riemannian manifold. Suppose that M is complete, has finite volume and admits an analytic and isometric $SL(3, \mathbb{R})$ -action with a dense orbit. If $8 < \dim(M) \leq 14$ and M is a weakly irreducible manifold, then \widetilde{M} is isomorphic to one of the following:*

- (1) $G_{2(2)}$.
- (2) $\mathbb{R} \backslash \widetilde{SL}(4, \mathbb{R})$.

The corresponding isomorphism is $\widetilde{SL}(3, \mathbb{R})$ -equivariant, where the $\widetilde{SL}(3, \mathbb{R})$ -action on it is induced by some non-trivial homomorphism of $\widetilde{SL}(3, \mathbb{R})$ into $G_{2(2)}$ or $\widetilde{SL}(4, \mathbb{R})$, respectively. We can also rescale the metric along $\widetilde{SL}(3, \mathbb{R})$ -orbits and their normal bundle to assume that such isomorphism is a local isometry for the bi-invariant pseudo-Riemannian metric on $G_{2(2)}$ or on $\mathbb{R} \backslash \widetilde{SL}(4, \mathbb{R})$ is given by the Killing form of $\mathfrak{g}_{2(2)}$ or $\mathfrak{sl}(4, \mathbb{R})$, respectively.

Additionally, if we assume that the universal covering of the manifold is not isometric to a quotient space then the option (1) can be eliminated in the previous theorem. This, $\widetilde{M} \cong G_{2(2)}$ and M is isometric to a quotient of $G_{2(2)}$ over a discrete subgroup. Which is the result of the next theorem.

Theorem C. *Let M be a connected analytic pseudo-Riemannian manifold. Suppose that M is complete, has finite volume and admits an analytic and isometric $SL(3, \mathbb{R})$ -action with a dense orbit. If $8 < \dim(M) \leq 14$ and M is a weakly irreducible manifold such that \widetilde{M} is not isomorphic to a quotient map then there exist*

- a lattice $\Gamma \subset G_{2(2)}$, and
- an analytic finite covering map $\varphi : G_{2(2)}/\Gamma \rightarrow M$

such that φ is $\widetilde{SL}(3, \mathbb{R})$ -equivariant map, where the $\widetilde{SL}(3, \mathbb{R})$ -action on $G_{2(2)}/\Gamma$ is induced by some non-trivial homomorphism $\widetilde{SL}(3, \mathbb{R}) \rightarrow G_{2(2)}$. We can also rescale the metric along $\widetilde{SL}(3, \mathbb{R})$ -orbits and their normal bundle to assume that φ is a local isometry for the bi-invariant pseudo-Riemannian metric on $G_{2(2)}$ given by the Killing form of its Lie algebra.

The proofs of Theorems A, B and C are based on the application of representation theory to the Killing vector fields that centralize the action of the group $\widetilde{SL}(3, \mathbb{R})$.

A fundamental result for this work is Proposition 2.2, which appears in [7] and [9] and whose detailed proof can be found in [10]. Such proposition shows the existence of a subalgebra of Killing vector fields on \widetilde{M} which vanish at some fixed point. The previous algebra is used to show properties of the centralizer of the $\widetilde{\mathrm{SL}}(3, \mathbb{R})$ -action, that we denote by \mathcal{H} . The analysis of such properties of \mathcal{H} lead us to identify the possible options of \widetilde{M} .

The organization of this paper is the following: In Section 1 we present the minimal dimension representation W of $\mathfrak{sl}(3, \mathbb{R})$ preserving a metric and therefore the decomposition of $\mathfrak{so}(W)$ as a direct sum of irreducible $\mathfrak{sl}(3, \mathbb{R})$ -modules. In Section 2, we show results that guarantee the existence of the centralizer of the action on the manifold M . In Section 3 we study the properties of the centralizer as a $\mathfrak{sl}(3, \mathbb{R})$ -module which lead us to obtain a lower bound of the dimension of M . In Section 4, we analyze the structure of the centralizer that permits to restrict the possibilities of the manifold M . Finally, in Section 5, we use the previous result to prove Theorems A, B and C.

1. AUTODUAL NON-TRIVIAL REPRESENTATIONS OF $\mathfrak{sl}(3, \mathbb{R})$

Recall that $\mathrm{SL}(3, \mathbb{R})$ denotes the **special linear group** of degree 3 over the field of real numbers. The group $\mathrm{SL}(3, \mathbb{R})$ is a connected, non-compact, real simple Lie group and Lie algebra denoted as $\mathfrak{sl}(3, \mathbb{R})$.

Let ρ be a representation of $\mathfrak{sl}(3, \mathbb{R})$ on the real vector space V_0 which induces a complex representation $\rho(\mathbb{C}) : \mathfrak{sl}(3, \mathbb{C}) \rightarrow V$, where $V = V_0 \otimes \mathbb{C}$. If a non-trivial representation ρ preserves a non-degenerated symmetric bilinear form then the representation $\rho(\mathbb{C})$ is *autodual*.

By the Weyl Character Formula, the lower dimensions of non-trivial irreducible representations of $\mathfrak{sl}(3, \mathbb{C})$ are 3, 6 and 8. Where 3 is the dimension of the natural representation of $\mathrm{SL}(3, \mathbb{C})$ onto \mathbb{C}^3 or onto its dual \mathbb{C}^{3*} , which are non-autodual. Therefore, one can see that the non-trivial representations of $\mathfrak{sl}(3, \mathbb{R})$ with dimension less than 6 are non-autodual.

The previous result is useful to show that the minimum dimension of a non-trivial representation of $\mathfrak{sl}(3, \mathbb{R})$ is 6, which is a corollary of the next lemma.

Lemma 1.1. *The minimal non-trivial real representation of $\mathfrak{sl}(3, \mathbb{R})$ preserving a non-degenerated bilinear symmetric form is isomorphic to $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$. That bilinear form has signature (3, 3).*

Proof. Let $\rho : \mathfrak{sl}(3, \mathbb{R}) \rightarrow V_0$ be a non-trivial real representation preserving a non-degenerated symmetric bilinear form $\langle \cdot, \cdot \rangle_0$. By the above we have that $\dim_{\mathbb{R}}(V_0) \geq 6$.

First, we assume that $\dim_{\mathbb{R}}(V_0) = 6$. If ρ is an irreducible representation then $\rho(\mathbb{C})$ is a complex irreducible representation. Therefore, Section 13 of [3] implies $V \simeq \mathrm{Sym}^2(\mathbb{C}^3)$ or $V \simeq \mathrm{Sym}^2(\mathbb{C}^{3*})$. In both cases, the representation is non-autodual.

On the other hand, if ρ is reducible (and therefore $\rho(\mathbb{C})$ is also reducible) then V is isomorphic to one of the following vector spaces: $\bigoplus_{j=1}^6 \mathbb{C}$, $\mathbb{C}^3 \oplus \bigoplus_{j=1}^3 \mathbb{C}$, $\mathbb{C}^{3*} \oplus \bigoplus_{j=1}^3 \mathbb{C}$, $\mathbb{C}^3 \oplus \mathbb{C}^3$, $\mathbb{C}^{3*} \oplus \mathbb{C}^{3*}$ or $\mathbb{C}^3 \oplus \mathbb{C}^{3*}$.

By the properties of representation of $\mathfrak{sl}(3, \mathbb{C})$ on \mathbb{C} , \mathbb{C}^3 and \mathbb{C}^{3*} we have that V cannot be isomorphic, as a $\mathfrak{sl}(3, \mathbb{C})$ -module, to $\bigoplus_{j=1}^6 \mathbb{C}$, $\mathbb{C}^3 \oplus \bigoplus_{j=1}^3 \mathbb{C}$, $\mathbb{C}^{3*} \oplus \bigoplus_{j=1}^3 \mathbb{C}$,

$\mathbb{C}^3 \oplus \mathbb{C}^3$ and $\mathbb{C}^{3*} \oplus \mathbb{C}^{3*}$. Hence, V_0 cannot be isomorphic (as $\mathfrak{sl}(3, \mathbb{R})$ -module) to $\bigoplus_{j=1}^6 \mathbb{R}, \mathbb{R}^3 \oplus \bigoplus_{j=1}^3 \mathbb{R}, \mathbb{R}^{3*} \oplus \bigoplus_{j=1}^3 \mathbb{R}, \mathbb{R}^3 \oplus \mathbb{R}^3$ and $\mathbb{R}^{3*} \oplus \mathbb{R}^{3*}$.

Next, we prove the existence of a non-degenerated symmetric bilinear form $\langle \cdot, \cdot \rangle_0$ in $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$ which is $\mathfrak{sl}(3, \mathbb{R})$ -invariant. Let $v, v' \in \mathbb{R}^3 \oplus \mathbb{R}^{3*}$ be given, then there exist unique elements $p, p' \in \mathbb{R}^3, q, q' \in \mathbb{R}^{3*}$ such that $v = (p, q)$ and $v' = (p', q')$. We define $\langle v, v' \rangle_0$ as follows

$$\langle v, v' \rangle_0 = \langle (p, q), (p', q') \rangle_0 := q(p') + q'(p),$$

where $q(p)$ is the evaluation of the element q in the vector p . Note that $\langle v, v' \rangle_0$ is a non-degenerated symmetric non-degenerated bilinear form. Let A be an arbitrary but fixed element of $\mathfrak{sl}(3, \mathbb{R})$ then,

$$\begin{aligned} \langle A \cdot v, v' \rangle + \langle v, A \cdot v' \rangle_0 &= \langle A \cdot (p, q), (p', q') \rangle_0 + \langle (p, q), A \cdot (p', q') \rangle_0 \\ &= \langle (A \cdot p, A \cdot q), (p', q') \rangle_0 + \langle (p, q), (A \cdot p', A \cdot q') \rangle_0 \\ &= (A \cdot q)(p') + q'(A \cdot p) + q(A \cdot p') + (A \cdot q')(p) \\ &= q(-A \cdot p') + q'(A \cdot p) + q(A \cdot p') + q'(-A \cdot p) \\ &= 0, \end{aligned}$$

which proves that $\langle \cdot, \cdot \rangle_0$ is $\mathfrak{sl}(3, \mathbb{R})$ -invariant.

Now, let $\langle \cdot, \cdot \rangle$ be a non-degenerated symmetric bilinear form on $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$ which is $\mathfrak{sl}(3, \mathbb{R})$ -invariant. If $\{e_1, e_2, e_3\}$ denotes the canonical basis of \mathbb{R}^3 then, given $i, j \in \{1, 2, 3\}$ such that $i \neq j$, we can find an element $A_{ij} \in \mathfrak{sl}(3, \mathbb{R})$ such that

$$A_{ij}(e_i) = e_i \quad \text{and} \quad A_{ij}(e_j) = -e_j.$$

The $\mathfrak{sl}(3, \mathbb{R})$ -invariance of $\langle \cdot, \cdot \rangle$ implies that

$$0 = \langle A_{ij}(e_i), e_j \rangle + \langle e_i, A_{ij}(e_j) \rangle = 2\langle e_i, e_j \rangle,$$

hence $\langle e_i, e_j \rangle = 0$. Thus, the signature of the bilinear form $\langle \cdot, \cdot \rangle_0$ on $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$ is $(3, 3)$. \square

Since the Lie algebra $\mathfrak{sl}(3, \mathbb{R})$ preserves a non-degenerate symmetric bilinear form on $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$ with signature $(3, 3)$, there exists a non-trivial Lie algebra homomorphism $\mathfrak{sl}(3, \mathbb{R}) \rightarrow \mathfrak{so}(3, 3)$. The simplicity of $\mathfrak{sl}(3, \mathbb{R})$ implies that such homomorphism is injective and therefore $\mathfrak{so}(3, 3)$ has a structure of $\mathfrak{sl}(3, \mathbb{R})$ -module.

Next, we analyze the decomposition of $\mathfrak{so}(3, 3)$ into a direct sum of irreducible $\mathfrak{sl}(3, \mathbb{R})$ -submodules.

From Table II in [1], $(\mathfrak{so}(3, 3), \mathfrak{gl}(3, \mathbb{R}))$ is a symmetric pair where $\mathfrak{gl}(3, \mathbb{R}) = \mathfrak{sl}(3, \mathbb{R}) \oplus \mathbb{R}$ is its decomposition as a direct sum of irreducible $\mathfrak{sl}(3, \mathbb{R})$ -modules.

Lemma 1.2. *$\mathfrak{so}(3, 3)$ is isomorphic, as a $\mathfrak{sl}(3, \mathbb{R})$ -module, to $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*} \oplus \mathbb{R}$.*

Proof. In general, Table II in [1], shows $(\mathfrak{so}(n, n), \mathfrak{sl}(n, \mathbb{R}) \oplus \mathbb{R})$ is a symmetric pair such that

$$\mathfrak{so}(n, n) = \mathfrak{sl}(n, \mathbb{R}) \oplus \mathbb{R} \oplus \nu_2(\mathfrak{sl}(3, \mathbb{R}))$$

where $\nu_2(\mathfrak{sl}(3, \mathbb{R}))$ is a self-adjoint $\mathfrak{sl}(n, \mathbb{R})$ -module containing $\pi_2(\mathfrak{sl}(n, \mathbb{R}))$, the irreducible representation of $\mathfrak{sl}(n, \mathbb{R})$ correspondent to its second fundamental weight ϖ_2 . Using a similar analysis, about self-adjoint representations, of Appendix A in [8] we have that $\nu_2(\mathfrak{sl}(3, \mathbb{R})) = \pi_2(\mathfrak{sl}(n, \mathbb{R})) \oplus \pi_{n-2}(\mathfrak{sl}(n, \mathbb{R}))$.

In the case $n = 3$ we have that $\pi_2(\mathfrak{sl}(3, \mathbb{R})) = \mathbb{R}^{3*}$ and $\pi_1(\mathfrak{sl}(3, \mathbb{R})) = \mathbb{R}^3$. Therefore

$$\mathfrak{so}(3, 3) = \mathfrak{sl}(3, \mathbb{R}) \oplus \mathbb{R} \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*}.$$

□

The following corollary is a consequence of the previous lemma.

Corollary 1.3. *The subalgebras of $\mathfrak{so}(3, 3)$ that are at the same time $\mathfrak{sl}(3, \mathbb{R})$ -submodules, with the structure of module induced by an injection of $\mathfrak{sl}(3, \mathbb{R})$ into $\mathfrak{so}(3, 3)$, are isomorphic to one of the following: 0 , $\mathfrak{sl}(3, \mathbb{R})$, \mathbb{R}^3 , \mathbb{R}^{3*} , \mathbb{R} , $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathbb{R}^3$, $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathbb{R}^{3*}$, $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathbb{R}$, $\mathbb{R}^3 \oplus \mathbb{R}$, $\mathbb{R}^{3*} \oplus \mathbb{R}$, $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}$, $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathbb{R}^{3*} \oplus \mathbb{R}$ or to $\mathfrak{so}(3, 3)$.*

Proof. We only recall that $(\mathfrak{so}(3, 3), \mathfrak{sl}(3, \mathbb{R}) \oplus \mathbb{R})$ is a symmetric pair. □

2. ISOMETRIC ACTIONS OF THE SIMPLE LIE GROUP $SL(3, \mathbb{R})$

In this section we assume G is a connected non-compact simple Lie group with Lie algebra \mathfrak{g} and M a connected and analytic finite-volume pseudo-Riemannian manifold where G acts analytic and isometrically with a dense orbit.

Since every isometric G -action on a manifold M with a dense orbit is locally free (see [11]), the orbits of the G -action define a foliation that we denote by \mathcal{F} . Then, for every $x \in M$ there exist a vector subspace of $T_x M$ such that $T_x M = T_x \mathcal{F} \oplus T_x \mathcal{F}^\perp$. The collection of these vector subspaces form a distribution on M that we will denote as $T\mathcal{F}^\perp$.

On the other hand, the tangent bundle to the foliation \mathcal{F} is a trivial vector bundle isomorphic to $M \times \mathfrak{g}$ under the isomorphism $M \times \mathfrak{g} \rightarrow T\mathcal{F}$, given by $(x, X) \mapsto X_x^*$. That also defines an isomorphism fiber of $T_x \mathcal{F}$ with \mathfrak{g} . For the rest of the paper, for an element X in the Lie algebra of a group acting on a manifold, we denote by X^* the vector field on the manifold whose one-parameter group of diffeomorphism is given by $(\exp(tX))_t$ through the action on the manifold.

The space of Killing fields of a geometric structure ω in a manifold M is denoted by $\text{Kill}(M, \omega)$, and $\text{Kill}_0(M, \omega, x)$ will denote the subspace of $\text{Kill}(M, \omega)$ consisting of vector fields that vanish on x . Here, we denote by σ the geometric structure of the pseudo-Riemannian metric on M . Unless otherwise is indicated, we will also omit the symbol that denotes the structure of pseudo-Riemannian metric, in particular, $\text{Kill}(M) := \text{Kill}(M, \sigma)$.

Let V be a vector space, we denote by $\mathfrak{so}(V)$ the Lie algebra of linear maps on V that are skew-symmetric with respect to a non-degenerate symmetric bilinear form on V .

As an immediate use of the previous definition and as consequence of the Jacobi identity we have the next lemma, showed in [7].

Lemma 2.1. *Let N be a pseudo-Riemannian manifold and $n \in N$. Then, the map $\lambda_n : \text{Kill}_0(N, n) \rightarrow \mathfrak{so}(T_n N)$ given by $\lambda_n(Z)(w) = [Z, W]_n$, where W is any vector field such that $W_n = w$, is a well defined homomorphism of Lie algebras.*

Here, the universal covering of any manifold N it will be denoted as \tilde{N} . Next, we present a result, proved in [10], which is fundamental in the present work.

Proposition 2.2 ([10, Proposition 2.3]). *Let G be a connected non-compact simple Lie group acting isometrically and with a dense orbit on a connected finite volume pseudo-Riemannian manifold M . Consider the \tilde{G} -action on \tilde{M} , lifted from the G -action on M . Assume that M and the G -action on M are both analytic. Then, there exists a conull subset $S \subset \tilde{M}$ such that for every $x \in S$ the following properties are satisfied:*

1. There is a homomorphism $\rho_x : \mathfrak{g} \rightarrow \text{Kill}(\widetilde{M})$ which is an isomorphism onto its image $\rho_x(\mathfrak{g}) = \mathfrak{g}(x)$.
2. $\mathfrak{g}(x) \subset \text{Kill}_0(\widetilde{M}, x)$, i.e. every element of $\mathfrak{g}(x)$ vanishes at x .
3. For every $X, Y \in \mathfrak{g}$ we have

$$[\rho_x(X), Y^*] = [X, Y]^* = -[X^*, Y^*].$$

In particular, the elements in $\mathfrak{g}(x)$ and their corresponding local flows preserve both \mathcal{F} and $T\mathcal{F}^\perp$.

4. The homomorphism of Lie algebras $\lambda_x \circ \rho_x : \mathfrak{g} \rightarrow \mathfrak{so}(T_x\widetilde{M})$ induces a \mathfrak{g} -module structure on $T_x\widetilde{M}$ for which subspaces $T_x\mathcal{F}$ and $T_x\mathcal{F}^\perp$ are \mathfrak{g} -submodules.

We consider the \mathfrak{g} -valued 1-form θ on \widetilde{M} which is defined, at every $x \in \widetilde{M}$, using the composition of the projection $T_x\widetilde{M} \rightarrow T_x\mathcal{F}$ and the isomorphism of $T_x\mathcal{F}$ with \mathfrak{g} . We also consider the \mathfrak{g} -valued 2-form given by $\Theta = d\theta|_{\wedge^2 T\mathcal{F}^\perp}$.

The proof of the following result can be found in [10, p. 239].

Lemma 2.3. *Let G , M and S be as in Proposition 2.2. If we assume that the G -orbits are non-degenerate, then:*

- (1) *For every $x \in S$, the maps $\theta_x : T_x\widetilde{M} \rightarrow \mathfrak{g}$ and $\Theta_x : \wedge^2 T_x\mathcal{F}^\perp \rightarrow \mathfrak{g}$ are both homomorphism of \mathfrak{g} -modules, for the \mathfrak{g} -module structures from Proposition 2.2.*
- (2) *The normal bundle $T\mathcal{F}^\perp$ is integrable if and only if $\Theta = 0$.*

Remark 2.4. Let G , M and S be as in Proposition 2.2, suppose that the G -orbits on M are non-degenerate. From the precedent Lemma and the analyticity of the elements involved in this work we have two possible cases: (i) $\Theta \equiv 0$ and then $T\mathcal{F}^\perp$ is integrable, or (ii) $\Theta \neq 0$ for almost $x \in \widetilde{M}$.

In what it follows we will assume that the G -orbits are non-degenerate with respect to the pseudo-Riemannian metric. Hence, the \widetilde{G} -orbits on \widetilde{M} are non-degenerate as well and we have a direct sum decomposition $T\widetilde{M} = T\mathcal{F} \oplus T\mathcal{F}^\perp$. The non-degeneracy of the orbits is ensured for manifolds of low dimension with respect to the dimension of the Lie group acting on this, that is the result of the following Lemma that can be founded in [10].

Lemma 2.5 ([10, Lemma 2.7]). *Let G be a connected non-compact simple Lie group acting isometrically and with a dense orbit on a connected finite volume pseudo-Riemannian manifold M . If $\dim(M) < 2 \dim(G)$, then the bundles $T\mathcal{F}$ and $T\mathcal{F}^\perp$ have fibers that are non-degenerated with respect to the metric on M .*

For the G -action as in Proposition 2.2, we consider \widetilde{M} endowed with the \widetilde{G} -action obtained by lifting the G -action on M . Let us denote by \mathcal{H} the Lie subalgebra of $\text{Kill}(\widetilde{M})$ consisting of the fields that centralize the \widetilde{G} -action on \widetilde{M} . Our first lemma involving \mathcal{H} is about an embedding of the Lie algebra \mathfrak{g} into \mathcal{H} . Such result allows us to apply representation theory to the study of \mathcal{H} .

Lemma 2.6 ([7, Lemma 1.7]). *Let S as in Proposition 2.2. Then, for every $x \in S$ and for ρ_x given as in Proposition 2.2, the map $\widehat{\rho}_x : \mathcal{H} \rightarrow \text{Kill}(\widetilde{M})$ defined as:*

$$\widehat{\rho}_x(X) = \rho_x(X) + X^*,$$

is an injective homomorphism of Lie algebras whose image $\mathcal{G}(x)$ lies in \mathcal{H} . In particular, $\hat{\rho}_x$ induces on \mathcal{H} a \mathfrak{g} -module structure such that $\mathcal{G}(x)$ is a submodule isomorphic to \mathfrak{g} .

Proof. First, observe that by the identity in Proposition 2.2(3), one can see that the image of $\hat{\rho}_x$ lies in \mathcal{H} .

To prove that $\hat{\rho}_x$ is a homomorphism of Lie algebras we apply Proposition 2.2 as follows: for $X, Y \in \mathfrak{g}$ we have

$$\begin{aligned} [\hat{\rho}_x(X), \hat{\rho}_x(Y)] &= [\rho_x(X) + X^*, \rho_x(Y) + Y^*] \\ &= [\rho_x(X), \rho_x(Y)] + [X, Y]^* + [X, Y]^* + [X^*, Y^*] \\ &= \rho_x([X, Y]) + [X, Y]^* \\ &= \hat{\rho}_x([X, Y]). \end{aligned}$$

From the definition of $\hat{\rho}_x$ we observe that $\hat{\rho}_x(X) = 0$ implies that $X^* = 0$, which in turns yields $X = 0$, because the G -action is locally free. Therefore, the last claim of our lemma is now clear. \square

The following lemma relates the structure of \mathfrak{g} -module of \mathcal{H} to that of $T_x \widetilde{M}$.

Lemma 2.7. *Let S as in Proposition 2.2. Consider $T_x \widetilde{M}$ and \mathcal{H} endowed with the \mathfrak{g} -module structure given by Proposition 2.2(4) and Lemma 2.6, respectively. Then, for almost every $x \in S$, the evaluation map:*

$$ev_x : \mathcal{H} \rightarrow T_x \widetilde{M}, \quad Z \mapsto Z_x$$

is a homomorphism of \mathfrak{g} -modules that satisfies $ev_x(\mathcal{G}(x)) = T_x \mathcal{F}$. Furthermore, for almost every $x \in S$ we have $ev_x(\mathcal{H}) = T_x \widetilde{M}$.

Proof. For every $x \in S$, if we let $Z \in \mathcal{H}$ and $X \in \mathfrak{g}$ be given, then:

$$ev_x(X \cdot Z) = [\hat{\rho}_x(X), Z]_x = [\rho_x(X) + X^*, Z]_x = [\rho_x(X), Z]_x = X \cdot Z_x = X \cdot ev_x(Z)$$

where we have used Lemma 2.1 and the definition of the \mathfrak{g} -module structures involved, thus proving the first part. The second part is proved by Lemma 4.1 in [13], using the transitivity of \mathfrak{h} on an open dense conull set in M . \square

The next lemma shows the existence of a relation between isometries and Killing fields for complete manifolds. It also shows that every Lie algebra containing Killing vector fields can be obtained from an isometric right action, and whose proof can be found in [7].

Lemma 2.8. *Let N be a complete pseudo-Riemannian manifold and H a simply connected Lie group with Lie algebra \mathfrak{h} . If $\psi : \mathfrak{h} \rightarrow \mathrm{Kill}(N)$ is a homomorphism of Lie algebras, then there exists an isometric right H -action, $N \times H \rightarrow N$, such that $\psi(X) = X^*$, for every $X \in \mathfrak{h}$. Furthermore, if N is analytic, then the H -action is analytic as well.*

3. ANALYSIS AND PROPERTIES OF ISOMETRIC $\mathrm{SL}(3, \mathbb{R})$ ACTIONS

In this section we assume that $G = \mathrm{SL}(3, \mathbb{R})$, which is a connected, non-compact simple Lie group, that the dimension of M satisfies that $8 < \dim(M) \leq 14$ and we remain the other hypotheses of Proposition 2.2. Hence, $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{R})$ and, from Lemma 2.5, we have that the orbits of the action on the manifold M are non-degenerated.

With our hypotheses in the previous paragraph we have the following result.

Theorem 3.1. *With the hypotheses in Theorem A, if we assume that our normal bundle $T\mathcal{F}^\perp$ is integrable then $\widetilde{M} \cong \widetilde{\mathrm{SL}}(3, \mathbb{R}) \times \widetilde{N}$, where \widetilde{N} is a complete pseudo-Riemannian manifold.*

Proof. This is a direct consequence of the main results in [10]. \square

The previous theorem induces to analyze the case when the normal bundle $T\mathcal{F}^\perp$ is not integrable. Hence, from now on we assume that the normal bundle to the foliation is not integrable.

Using the analyticity of our hypotheses we have the following result.

Lemma 3.2. *Let S as in Proposition 2.2. Let $x \in S$ be, consider $T_x\mathcal{F}^\perp$ endowed with the $\mathfrak{sl}(3, \mathbb{R})$ -module structure given by Proposition 2.2(4). Then, for almost every $x \in S$, $T_x\mathcal{F}^\perp$ is isomorphic to $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$ and $\dim(M) = 14$. In particular, the algebra $\mathfrak{so}(T_x\mathcal{F}^\perp)$ is isomorphic to $\mathfrak{so}(3, 3)$ as a Lie algebra and as a $\mathfrak{sl}(3, \mathbb{R})$ -module.*

Proof. By the non-integrability of $T\mathcal{F}^\perp$ then, from Lemma 2.3(2), we have that the 2-form Θ is not equal to zero. Since this 2-form is analytic, then it vanishes on a proper analytic subset of null measure. Hence, $\Theta_x \neq 0$ for almost every $x \in S$.

Choose and fix an element $x \in S$ such that $\Theta_x \neq 0$. By the definition of Θ_x we have that $T_x\mathcal{F}^\perp$ is a non-trivial $\mathfrak{sl}(3, \mathbb{R})$ -module preserving a non-degenerate symmetric bilinear form with $1 < \dim(T_x\mathcal{F}^\perp) \leq 6$, then from Lemma 1.1 and Proposition 2.2(4), $T_x\mathcal{F}^\perp \simeq \mathbb{R}^3 \oplus \mathbb{R}^{3*}$ as a $\mathfrak{sl}(3, \mathbb{R})$ -module. Hence, $\dim(T_x\mathcal{F}^\perp) = 6$ and, therefore, $\dim(M) = 14$.

On the other hand, by Lemma 1.1, the representation of $\mathfrak{sl}(3, \mathbb{R})$ on $T_x\mathcal{F}^\perp$ defines a non-trivial homomorphism of Lie algebras $\mathfrak{so}(T_x\mathcal{F}^\perp) \rightarrow \mathfrak{so}(3, 3)$, which is also an isomorphism of $\mathfrak{sl}(3, \mathbb{R})$ -modules. Since $\mathfrak{so}(3, 3)$ is a simple Lie algebra, this latter homomorphism is injective and so is an isomorphism. \square

The results of the previous lemma allow us to obtain a decomposition of the centralizer \mathcal{H} of the $\widetilde{\mathrm{SL}}(3, \mathbb{R})$ -action into submodules related to the pseudo-Riemannian metric structure on \widetilde{M} . First, recall that Lemma 2.6 induces on \mathcal{H} a structure of $\mathfrak{sl}(3, \mathbb{R})$ -module.

Lemma 3.3. *Let S be as in Proposition 2.2. Then, for almost every $x \in S$ there is a decomposition of \mathcal{H} into $\mathfrak{sl}(3, \mathbb{R})$ -submodules, $\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{W}(x)$, satisfying:*

- 1) $\mathcal{G}(x) = \widehat{\rho}_x(\mathfrak{sl}(3, \mathbb{R}))$ is a Lie subalgebra of \mathcal{H} isomorphic to $\mathfrak{sl}(3, \mathbb{R})$ and $ev_x(\mathcal{G}(x)) = T_x\mathcal{F}$.
- 2) $\mathcal{H}_0(x) = \ker(ev_x)$ is a Lie subalgebra and a $\mathfrak{sl}(3, \mathbb{R})$ -module of \mathcal{H} , isomorphic to a subset of $\mathfrak{so}(T_x\mathcal{F}^\perp)$.
- 3) $\mathcal{W}(x)$ is isomorphic to $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$ as $\mathfrak{sl}(3, \mathbb{R})$ -module and $ev_x(\mathcal{W}(x)) = T_x\mathcal{F}^\perp$.

In particular, the evaluation map (at x) defines an isomorphism of $\mathfrak{sl}(3, \mathbb{R})$ -modules $\mathcal{G}(x) \oplus \mathcal{W}(x) \rightarrow T_x\mathcal{F} \oplus T_x\mathcal{F}^\perp$, preserving the summands in that order. In 2), we have that the considered isomorphism is as a Lie algebra and as an $\mathfrak{sl}(3, \mathbb{R})$ -module.

Proof. Let us choose and fix an element $x \in S$ that satisfies Lemma 2.7 and Lemma 3.2. By Lemma 2.6 we conclude that $\mathcal{G}(x) = \widehat{\rho}_x(\mathfrak{sl}(3, \mathbb{R}))$ is a Lie algebra isomorphic to $\mathfrak{sl}(3, \mathbb{R})$.

Define $\mathcal{H}_0(x) = \ker(ev_x)$. By Lemma 2.7, it follows that $\mathcal{H}_0(x)$ is an $\mathfrak{sl}(3, \mathbb{R})$ -submodule of \mathcal{H} . On the other hand, since $\mathcal{H}_0(x) = \mathcal{H} \cap \mathrm{Kill}_0(\widetilde{M}, x)$, one can see that $\mathcal{H}_0(x)$ is a Lie subalgebra of \mathcal{H} .

Since the elements of $\mathcal{G}(x)$ are of the form $\rho_x(X) + X^*$, with $X \in \mathfrak{sl}(3, \mathbb{R})$. Hence, for any such element we have $ev_x(\rho_x(X) + X^*) = X_x^*$. That and the condition $ev_x(\rho_x(X) + X^*) = 0$ imply that $X = 0$. In other words, $\mathcal{G}(x) \cap \mathcal{H}_0(x) = 0$; therefore there exists an $\mathfrak{sl}(3, \mathbb{R})$ -submodule complementary $\mathcal{W}'(x)$ to $\mathcal{G}(x) \oplus \mathcal{H}_0(x)$ in \mathcal{H} . Note, we have an isomorphism from $\mathcal{G}(x) \oplus \mathcal{W}'(x)$ onto $T_x \widetilde{M}$ via the evaluation map. We choose $\mathcal{W}(x)$ as the inverse image of $T_x \mathcal{F}^\perp$ under our previous isomorphism. We have our desired decomposition of \mathcal{H} into $\mathfrak{sl}(3, \mathbb{R})$ -submodules.

Let $\mathrm{Kill}_0(\widetilde{M}, x, \mathcal{F})$ be the Lie algebra of Killing vector fields on \widetilde{M} which preserves the foliation \mathcal{F} and vanish at x . Note that every vector field in $\mathrm{Kill}_0(\widetilde{M}, x, \mathcal{F})$ leaves invariant the normal bundle. On the other hand, the map λ_x restricted to $\mathrm{Kill}_0(\widetilde{M}, x, \mathcal{F})$ induces the following homomorphism of Lie algebras:

$$\lambda_x^\perp : \mathrm{Kill}_0(\widetilde{M}, x, \mathcal{F}) \rightarrow \mathfrak{so}(T_x \mathcal{F}^\perp), \quad X \mapsto \lambda_x(X)|_{T_x \mathcal{F}^\perp}.$$

Observe that both $\rho_x(\mathfrak{sl}(3, \mathbb{R}))$ and $\mathcal{H}_0(x)$ lie inside of $\mathrm{Kill}_0(\widetilde{M}, x, \mathcal{F})$.

Claim 1: λ_x^\perp is injective when is restricted to $\mathfrak{sl}(3, \mathbb{R})(x)$. By our choice of the element $x \in S$ and the results in Lemma 3.2, the map $\lambda_x^\perp \circ \rho_x : \mathfrak{sl}(3, \mathbb{R}) \rightarrow \mathfrak{so}(T_x \mathcal{F}^\perp)$ is a non-trivial homomorphism of Lie algebras. Since $\mathfrak{sl}(3, \mathbb{R})$ is a simple Lie algebra then the map λ_x^\perp restricted to $\mathfrak{sl}(3, \mathbb{R})$ is injective.

Claim 2: λ_x^\perp restricted to $\mathcal{H}_0(x)$ is injective. Recall that pseudo-Riemannian metric structures are 1-rigid structures (see [2]), therefore a Killing vector space is completely determined by its 1-jet at x . If $Z \in \mathcal{H}_0(x)$ is given, then $Z_x = ev_x(Z) = 0$; so it is determined by the values $[Z, V]_x$, for V vector field on a neighborhood of x . Since Z is in the centralizer of the $\widetilde{\mathrm{SL}}(3, \mathbb{R})$ -action, then $[Z, X^*]_x = 0$ for all $X \in \mathfrak{sl}(3, \mathbb{R})$, so $[Z, V]_x = 0$ when $V_x \in T_x \mathcal{F}$. Hence, if $[Z, V]_x = 0$ when $V_x \in T_x \mathcal{F}^\perp$, this implies that $Z = 0$. Therefore, we have that λ_x^\perp is injective when it is restricted to $\mathcal{H}_0(x)$.

On the other hand, if $X \in \mathfrak{sl}(3, \mathbb{R})$ and $Y \in \mathcal{H}_0(x)$ then

$$\begin{aligned} \lambda_x^\perp(X \cdot Y) &= \lambda_x^\perp([\widehat{\rho}_x(X), Y]) = \lambda_x^\perp([\rho_x(X) + X^*, Y]) \\ &= \lambda_x^\perp([\rho_x(X), Y]) = [\lambda_x^\perp(\rho_x(X)), \lambda_x^\perp(Y)] \\ &= X \cdot \lambda_x^\perp(Y). \end{aligned}$$

That shows that the map λ_x^\perp restricted to $\mathcal{H}_0(x)$ is a homomorphism of $\mathfrak{sl}(3, \mathbb{R})$ -modules. \square

Remark 3.4. By Lemma 3.3 we have that $\mathcal{H}_0(x)$ is a subalgebra, and an $\mathfrak{sl}(3, \mathbb{R})$ -submodule, isomorphic to $\lambda_x^\perp(\mathcal{H}_0(x)) \subset \mathfrak{so}(T_x \mathcal{F}^\perp)$. On the other hand, since $\mathfrak{so}(T_x^\perp)$ is isomorphic to $\mathfrak{so}(3, 3)$ (and hence to $\mathfrak{sl}(4, \mathbb{R})$), then $\mathcal{H}_0(x)$ is isomorphic to one of the Lie subalgebras in Corollary 1.3.

By Lemma A.5 in [8], we have that $\wedge^2 T_x \mathcal{F}^\perp$ is isomorphic to $\mathfrak{so}(T_x \mathcal{F}^\perp)$ as $\mathfrak{so}(T_x \mathcal{F}^\perp)$ -module. Thus, from the definition of the map Θ_x in Lemma 2.3 and Lemma A.5 in [8], we can consider Θ_x as a map from $\mathfrak{so}(T_x \mathcal{F}^\perp)$ to $\mathfrak{sl}(3, \mathbb{R})$.

More properties about the subalgebra $\mathcal{H}_0(x)$ can be obtained considering the map Θ_x as in the previous paragraph, one of these is contained in the following result which appears in Proposition 3.10 and Proposition 3.11 in [8].

Theorem 3.5. *For almost every $x \in S$ as in Lemma 3.3. $\lambda_x^\perp(\mathcal{H}_0(x))$ is a $\lambda_x^\perp(\mathcal{G}(x))$ -submodule and a Lie algebra of $\mathfrak{so}(T_x \mathcal{F}^\perp)$ that satisfies*

$$[\lambda_x^\perp(\mathcal{H}_0(x)), \mathfrak{so}(T_x \mathcal{F}^\perp)] \subset \ker(\Theta_x).$$

Proof. Let $x \in S$ be an element satisfying Lemma 3.3, so it does Lemma 2.7 and Lemma 3.2.

The element $x \in S$ has the required properties of the hypotheses of Proposition 3.10 in [8]. Therefore, the proof of this theorem is similar to that of the above mentioned proposition. \square

Remark 3.6. Decomposing $\mathfrak{so}(T_x \mathcal{F}^\perp)$ (isomorphic to $\mathfrak{so}(3, 3)$) as a direct sum of irreducible $\mathfrak{sl}(3, \mathbb{R})$ -submodules, and its corresponding bracket product, Lemma 1.2 yields that the only possibilities for $\mathcal{H}_0(x)$ satisfying Theorem 3.5 are 0, or isomorphic to \mathbb{R} .

4. STRUCTURE OF THE CENTRALIZER AND ITS CONSEQUENCES

The previous section was devoted to show all the possible values that $\mathcal{H}_0(x)$ can take. This section we analyze the implications of all these possible cases. Here, we assume the same hypotheses and notation of Lemma 3.3.

First, fixed an arbitrarily element $x \in S$ as in Lemma 3.3 and Theorem 3.5, guaranty (by Lemma 3.2 and Lemma 3.3) we can choose subspaces $\mathcal{V}(x)$ and $\mathcal{V}^*(x)$ of $\mathcal{W}(x)$ such that $\mathcal{W}(x) = \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ and $\mathcal{G}(x)$ acts on $\mathcal{V}(x)$ (resp. $\mathcal{V}^*(x)$) as $\mathfrak{sl}(3, \mathbb{R})$ acts on \mathbb{R}^3 (resp. \mathbb{R}^{3*}).

Thus, \mathcal{H} is an $\mathfrak{sl}(3, \mathbb{R})$ -module and on properties of the evaluation map in Lemma 3.3, we have the following properties:

$$(4.1) \quad [\mathcal{G}(x), \mathcal{H}_0(x)] \subseteq \mathcal{H}_0(x)$$

$$(4.2) \quad [\mathcal{G}(x), \mathcal{W}(x)] = \mathcal{W}(x)$$

$$(4.3) \quad [\mathcal{H}_0(x), \mathcal{W}(x)] \subseteq \mathcal{H}_0(x) \oplus \mathcal{W}(x)$$

$$(4.4) \quad [\mathcal{H}_0(x), \mathcal{H}_0(x)] \subseteq \mathcal{H}_0(x)$$

In particular, when $\mathcal{H}_0(x)$ is isomorphic to \mathbb{R} or to 0, the contention (4.3) is strict. That is a consequence of the following lemma.

Lemma 4.1. *As in Lemma 3.3, if $\mathcal{H}_0(x)$ is isomorphic either to \mathbb{R} , $\mathfrak{sl}(3, \mathbb{R})$, $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathbb{R}$ or $\mathfrak{sl}(4, \mathbb{R})$ then $[\mathcal{H}_0(x), \mathcal{W}(x)] = \mathcal{W}(x)$.*

Next, we analyze the different possibilities of $\mathcal{H}_0(x)$ satisfying Theorem 3.5. Recall that $\mathcal{H}_0(x)$ is isomorphic to one of the subalgebras of Remark 3.6.

4.0.1. $\mathcal{H}_0(x) = 0$.

Lemma 4.2. *Let S be as in Lemma 3.3. If $x \in S$ and $\mathcal{H}_0(x) = 0$ then, one of the following occurs:*

- (1) *The Radical of \mathcal{H} is $\mathcal{W}(x) = \mathcal{V}(x) \oplus \mathcal{V}^*(x)$.*
- (2) *$\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is isomorphic to $\mathfrak{g}_{2(2)}$.*

Proof. Let us choose an arbitrary but fixed element $x \in S$ such that $\mathcal{H}_0(x) = 0$, as in Lemma 3.3. Hence, $\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{W}(x) = \mathcal{G}(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$.

Since $\mathcal{G}(x)$ is a simple Lie algebra, we can choose \mathfrak{s} a Levi factor of \mathcal{H} which contains $\mathcal{G}(x)$. Since $\mathcal{G}(x) \subseteq \mathfrak{s}$ and considering the structure of \mathcal{H} as an $\mathfrak{sl}(3, \mathbb{R})$ -module, then \mathfrak{s} is also an $\mathfrak{sl}(3, \mathbb{R})$ -module.

Let W be an $\mathfrak{sl}(3, \mathbb{R})$ -module of \mathcal{H} such that $\mathfrak{s} = \mathcal{G}(x) \oplus W$. Moreover, since $\text{rad}(\mathcal{H})$ is an ideal of \mathcal{H} this induces the decomposition of \mathcal{H} as a direct sum of $\mathfrak{sl}(3, \mathbb{R})$ -modules:

$$\mathcal{H} = \mathfrak{s} \oplus \text{rad}(\mathcal{H}) = \mathcal{G}(x) \oplus W \oplus \text{rad}(\mathcal{H})$$

that we compare with its decomposition into irreducible $\mathfrak{sl}(3, \mathbb{R})$ -submodules from Lemma 3.3

$$\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x).$$

The properties of representations of Lie algebras and the decompositions of \mathcal{H} imply that one of the following must occur:

- (a) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{V}(x)$ and $\text{rad}(\mathcal{H}) = \mathcal{V}^*(x)$.
- (b) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{V}^*(x)$ and $\text{rad}(\mathcal{H}) = \mathcal{V}(x)$.
- (c) $\mathfrak{s} = \mathcal{G}(x)$ and $\text{rad}(\mathcal{H}) = \mathcal{V}(x) \oplus \mathcal{V}^*(x)$.
- (d) $\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is semisimple.

Suppose the case (a) is satisfied:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{V}(x) \quad \text{and} \quad \text{rad}(\mathcal{H}) = \mathcal{V}^*(x).$$

Because $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{V}(x)$ is a semisimple Lie algebra, then \mathfrak{s} is a finite direct product, of simple ideals $\mathfrak{h}_1 \times \mathfrak{h}_2 \times \cdots \times \mathfrak{h}_k$. Since, every ideal is invariant by $\mathcal{G}(x)$ then these ideals are $\mathfrak{sl}(3, \mathbb{R})$ -modules. Representation properties of $\mathfrak{sl}(3, \mathbb{R})$ and the decomposition of \mathfrak{s} in a direct sum of irreducible $\mathfrak{sl}(3, \mathbb{R})$ -modules yield that $k \leq 2$.

If $k = 2$, $\mathfrak{s} = \mathfrak{h}_1 \times \mathfrak{h}_2$. By (4.2) we have that $[\mathcal{G}(x), \mathcal{V}(x)] \subseteq \mathcal{V}(x)$. Additionally, since $\mathcal{V}(x)$ is isomorphic to \mathbb{R}^3 as $\mathfrak{sl}(3, \mathbb{R})$ -module then, $[\mathcal{V}(x), \mathcal{V}(x)] \neq 0$ implies that $[\mathcal{V}(x), \mathcal{V}(x)] \simeq \mathbb{R}^{3*}$. Which shows that $[\mathcal{V}(x), \mathcal{V}(x)] = 0$. Therefore $\mathcal{V}(x)$ is an ideal of $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{V}(x)$. Without loss of generality, we assume $\mathfrak{h}_2 = \mathcal{V}(x)$, then the simple ideal \mathfrak{h}_2 must be an abelian ideal, thus $k = 1$.

However, if $k = 1$, \mathfrak{s} is a simple Lie algebra. Thence, $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{V}(x)$ is a 11-dimensional real simple Lie algebra. Therefore, $\mathfrak{s}^{\mathbb{C}}$ is a complex simple Lie algebra with complex dimension 11 and [4, p. 516], showed it cannot happen. Then, we have proved that case (a) cannot happen.

Assume case (b) is satisfied:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{V}^*(x) \quad \text{and} \quad \text{rad}(\mathcal{H}) = \mathcal{V}(x).$$

This case is ruled out by a similar argument of case (a).

Suppose case (c) is satisfied:

$$\mathfrak{s} = \mathcal{G}(x) \quad \text{and} \quad \text{rad}(\mathcal{H}) = \mathcal{V}(x) \oplus \mathcal{V}^*(x).$$

Since $\mathcal{V}(x) \simeq \mathbb{R}^3$ and $\mathcal{V}^*(x) \simeq \mathbb{R}^{3*}$ as $\mathfrak{sl}(3, \mathbb{R})$ -modules then by properties of representation of $\mathfrak{sl}(3, \mathbb{R})$ we have that $[\mathcal{V}(x), \mathcal{V}^*(x)]$ is isomorphic to a subspace of $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathbb{R}$ as a $\mathfrak{sl}(3, \mathbb{R})$ -module. Therefore, we have that $[\mathcal{V}(x), \mathcal{V}(x)] \subseteq \mathcal{V}^*(x)$, $[\mathcal{V}(x), \mathcal{V}^*(x)] = 0$ and $[\mathcal{V}^*(x), \mathcal{V}^*(x)] \subseteq \mathcal{V}(x)$. And by the solvability of $\text{rad}(\mathcal{H})$ we have that $[\mathcal{V}(x), \mathcal{V}(x)] = 0$ or $[\mathcal{V}^*(x), \mathcal{V}^*(x)] = 0$.

If case (d) is satisfied:

$$\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x) \quad \text{is a simple Lie algebra.}$$

Using the same argument as in case (a), \mathcal{H} is a direct product of a finite number of simple ideals $\mathfrak{h}_1 \times \mathfrak{h}_2 \times \cdots \times \mathfrak{h}_k$ where every ideal is an $\mathfrak{sl}(3, \mathbb{R})$ -module and $k \leq 3$.

If $k = 3$, $\mathcal{H} = \mathfrak{h}_1 \times \mathfrak{h}_2 \times \mathfrak{h}_3$, we can assume, reindexing if necessary, that $\mathfrak{h}_3 = \mathcal{V}^*(x)$ and $\mathfrak{h}_1 \times \mathfrak{h}_2 = \mathcal{G}(x) \oplus \mathcal{V}(x)$. Then $[\mathcal{V}^*(x), \mathcal{V}^*(x)] \subseteq \mathfrak{h}_3$, but similar to case (a) we have that $[\mathcal{V}^*(x), \mathcal{V}^*(x)] = 0$, so \mathfrak{h}_3 is an abelian Lie algebra and this is not possible. Therefore $k \leq 2$.

If $k = 2$, $\mathcal{H} = \mathfrak{h}_1 \times \mathfrak{h}_2$, after decomposing \mathfrak{h}_1 and \mathfrak{h}_2 as the direct sum of $\mathfrak{sl}(3, \mathbb{R})$ -modules, and reindexing if necessary, we can assume that \mathfrak{h}_1 is an irreducible $\mathfrak{sl}(3, \mathbb{R})$ -module and $\mathfrak{h}_2 = V_1 \oplus V_2$, where V_1 and V_2 are irreducible $\mathfrak{sl}(3, \mathbb{R})$ -modules. We can also assume that $\mathcal{V}^*(x) \subset \mathfrak{h}_2$ and $\mathcal{V}(x) \oplus \mathcal{V}^*(x) = \mathfrak{h}_2$, because as in the previous case, it cannot happen that $\mathfrak{h}_1 = \mathcal{V}(x)$. Then $\mathcal{G}(x) \subseteq \mathfrak{h}_1$ and $[\mathcal{G}(x), \mathcal{W}(x)] = [\mathfrak{h}_1, \mathfrak{h}_2] = 0$ that contradicts the equation (4.2).

If $k = 1$, $\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is a real simple Lie algebra of dimension 14. Then, \mathcal{H} is the realification of a complex simple Lie algebra of dimension 7 or its complexification, $\mathcal{H}^{\mathbb{C}}$, is a complex simple Lie algebra. Since, by [4, p. 516], there is not a complex simple Lie algebra of dimension 7. So, $\mathcal{H}^{\mathbb{C}}$ is a complex simple Lie algebra with $\dim_{\mathbb{C}}(\mathcal{H}^{\mathbb{C}}) = 14$. Then, $\mathcal{H}^{\mathbb{C}} \cong \mathfrak{g}_2$. From here, \mathcal{H} is isomorphic to a real form of \mathfrak{g}_2 .

On the other hand, we recall that \mathcal{H} contains a Lie subalgebra isomorphic to $\mathfrak{sl}(3, \mathbb{R})$, that is simple and non-compact. Then, \mathcal{H} is non-compact. Otherwise, exercise 4(ii) in the page 152 of [4], would imply that $\mathfrak{sl}(3, \mathbb{R})$ is compact, which is a contradiction.

Since, by [4, p. 518], there is only a non-compact real form of \mathfrak{g}_2 , namely $\mathfrak{g}_{2(2)}$. Then

$$\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x) \simeq \mathfrak{g}_{2(2)}.$$

□

4.0.2. $\mathcal{H}_0(x) \simeq \mathbb{R}$.

Lemma 4.3. *Let S be as in Lemma 3.3. Let $x \in S$ be such that $\mathcal{H}_0(x)$ is isomorphic to \mathbb{R} as an $\mathfrak{sl}(3, \mathbb{R})$ -module then one of the following occurs*

- (1) *The radical of \mathcal{H} is $\mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ where $\mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is a Lie subalgebra.*
- (2) *$\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is a simple Lie algebra isomorphic to $\mathfrak{sl}(4, \mathbb{R})$.*

Proof. Let us choose an arbitrary but fixed element $x \in S$, as in Lemma 3.3, such that $\mathcal{H}_0(x) \simeq \mathbb{R}$.

Choose \mathfrak{s} a Levi factor of \mathcal{H} that contains $\mathcal{G}(x)$. Similar to the case $\mathcal{H}_0(x) = 0$, \mathfrak{s} is a $\mathfrak{sl}(3, \mathbb{R})$ -submodule of \mathcal{H} .

Let W be a $\mathfrak{sl}(3, \mathbb{R})$ -submodule of \mathcal{H} such that $\mathfrak{s} = \mathcal{G}(x) \oplus W$. Since $\text{rad}(\mathcal{H})$ is an ideal, this induces the next decomposition of \mathcal{H} as a direct sum of $\mathfrak{sl}(3, \mathbb{R})$ -modules:

$$\mathcal{H} = \mathcal{G}(x) \oplus W \oplus \text{rad}(\mathcal{H})$$

that we compare with the decomposition of irreducible $\mathfrak{sl}(3, \mathbb{R})$ -modules in Lemma 3.3

$$\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x).$$

By the properties of representations of Lie algebras and the decomposition of $\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$, one of the following must occur:

- (a) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x)$ and $\text{rad}(\mathcal{H}) = \mathcal{V}^*(x)$.
- (b) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}^*(x)$ and $\text{rad}(\mathcal{H}) = \mathcal{V}(x)$.
- (c) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ and $\text{rad}(\mathcal{H}) = \mathcal{H}_0(x)$.
- (d) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{H}_0(x)$ and $\text{rad}(\mathcal{H}) = \mathcal{V}(x) \oplus \mathcal{V}^*(x)$.
- (e) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{V}(x)$ and $\text{rad}(\mathcal{H}) = \mathcal{H}_0(x) \oplus \mathcal{V}^*(x)$.
- (f) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{V}^*(x)$ and $\text{rad}(\mathcal{H}) = \mathcal{H}_0(x) \oplus \mathcal{V}(x)$.
- (g) $\mathfrak{s} = \mathcal{G}(x)$ and $\text{rad}(\mathcal{H}) = \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$.

(h) $\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is semisimple.

Suppose that case (a) is satisfied:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x) \quad \text{and} \quad \text{rad}(\mathcal{H}) = \mathcal{V}^*(x).$$

Recall that $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x)$ is a semisimple Lie algebra.

Since $\mathcal{V}(x) \simeq \mathbb{R}^3$ and $[\mathcal{V}(x), \mathcal{V}(x)] \subseteq \mathfrak{s}$. If $[\mathcal{V}(x), \mathcal{V}(x)] \neq 0$ then $[\mathcal{V}(x), \mathcal{V}(x)]$ is isomorphic to \mathbb{R}^{3*} . Therefore, the projection of $[\mathcal{V}(x), \mathcal{V}(x)]$ in $\mathcal{G}(x)$, $\mathcal{H}_0(x)$ and $\mathcal{V}(x)$ is 0. That implies that $[\mathcal{V}(x), \mathcal{V}(x)] = 0$. Hence, $\mathcal{V}(x)$ is an abelian ideal of \mathfrak{s} , which is a contradiction. So, case (a) cannot be possible.

Case (b) is not possible and the proof is similar to (a).

Now suppose case (c) is satisfied:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x) \quad \text{and} \quad \text{rad}(\mathcal{H}) = \mathcal{H}_0(x).$$

Here, by Lemma 4.1, we have that $[\mathcal{H}_0(x), \mathcal{V}(x) \oplus \mathcal{V}^*(x)] = \mathcal{V}(x) \oplus \mathcal{V}^*(x)$. On the other hand, since $\text{rad}(\mathcal{H})$ is an ideal of \mathcal{H} , then $\mathcal{V}(x) \oplus \mathcal{V}^*(x) \subseteq \text{rad}(\mathcal{H})$. So, this case cannot occur.

Suppose case (d) is satisfied:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \quad \text{and} \quad \text{rad}(\mathcal{H}) = \mathcal{V}(x) \oplus \mathcal{V}^*(x).$$

From (4.1) and (4.4), we have that $\mathcal{H}_0(x)$ is an abelian ideal of \mathfrak{s} . Which is a contradiction. Then case (d) is not possible.

Suppose, the case (e) is satisfied:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{V}(x) \quad \text{and} \quad \text{rad}(\mathcal{H}) = \mathcal{H}_0(x) \oplus \mathcal{V}^*(x).$$

Similar to the argument in case (a), $\mathcal{V}(x)$ is an abelian ideal of \mathfrak{s} . Therefore, this case is not possible.

Case (f),

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{V}^*(x) \quad \text{and} \quad \text{rad}(\mathcal{H}) = \mathcal{H}_0(x) \oplus \mathcal{V}(x),$$

cannot happen and the proof is similar to that of (e).

Now, we suppose (g) is satisfied:

$$\mathfrak{s} = \mathcal{G}(x) \quad \text{and} \quad \text{rad}(\mathcal{H}) = \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x).$$

By Lemma 4.1 we have $[\mathcal{H}_0(x), \mathcal{W}(x)] = \mathcal{W}(x)$. Let $\pi_0 : \text{rad}(\mathcal{H}) \rightarrow \mathcal{H}_0(x)$ be the projection map on the first component of $\text{rad}(\mathcal{H})$, since $[\mathcal{W}(x), \mathcal{W}(x)] \subseteq \text{rad}(\mathcal{H})$ then $\pi_0([\mathcal{W}(x), \mathcal{W}(x)]) = 0$. Otherwise $\text{rad}(\mathcal{H})$ will not be solvable. In conclusion, $\mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is a Lie subalgebra of $\text{rad}(\mathcal{H})$.

If case (h) is satisfied:

$$\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x) \quad \text{is semisimple.}$$

Here, \mathcal{H} is a finite direct product $\mathfrak{h}_1 \times \mathfrak{h}_2 \times \cdots \times \mathfrak{h}_k$ of simple ideals which also are $\mathfrak{sl}(3, \mathbb{R})$ -modules with $k \leq 4$.

If $k = 4$, $\mathcal{H} = \mathfrak{h}_1 \times \mathfrak{h}_2 \times \mathfrak{h}_3 \times \mathfrak{h}_4$. Without loss of generality we can assume that $\mathfrak{h}_4 = \mathcal{H}_0(x)$ is a simple ideal of \mathcal{H} . Since $\mathcal{H}_0(x) \simeq \mathbb{R}$ is abelian, this is a contradiction. Therefore $k = 4$ cannot be possible and $k \leq 3$.

If $k = 3$, $\mathcal{H} = \mathfrak{h}_1 \times \mathfrak{h}_2 \times \mathfrak{h}_3$. Suppose, reindexing if necessary, that \mathfrak{h}_1 and \mathfrak{h}_2 are irreducible $\mathfrak{sl}(3, \mathbb{R})$ -modules and $\mathfrak{h}_3 = V_1 \oplus V_2$, where V_1 and V_2 are irreducibles $\mathfrak{sl}(3, \mathbb{R})$ -modules. We can also assume that $\mathcal{H}_0(x) \subset \mathfrak{h}_3$ then, by Lemma 4.1, $\mathcal{V}(x) \oplus \mathcal{V}^*(x) \subset \mathfrak{h}_3$. That implies that \mathfrak{h}_1 or \mathfrak{h}_2 is equal to 0. Therefore, this case is not possible and $k \leq 2$.

If $k = 2$, $\mathcal{H} = \mathfrak{h}_1 \times \mathfrak{h}_2$. As in case $k = 3$, assume $\mathcal{H}_0(x) \oplus \mathcal{W}(x) \subset \mathfrak{h}_2$. On the other hand, by (4.2), $[\mathcal{G}(x), \mathcal{W}(x)] = \mathcal{W}(x)$ then $\mathcal{G}(x) \subset \mathfrak{h}_2$ and $\mathfrak{h}_1 = 0$, which is a contradiction. Hence, this case cannot be possible and $k = 1$.

If $k = 1$, \mathcal{H} is a real simple Lie algebra of dimension 15. Therefore, $\mathcal{H}^\mathbb{C}$ is a complex simple Lie algebra with $\dim_\mathbb{C}(\mathcal{H}^\mathbb{C}) = 15$. Then, by [4, p. 516], $\mathcal{H}^\mathbb{C} \simeq \mathfrak{sl}(4, \mathbb{C})$. So, \mathcal{H} is isomorphic to a non-compact real form of $\mathfrak{sl}(4, \mathbb{C})$.

From [4, Table V], the only non-compact real forms of $\mathfrak{sl}(4, \mathbb{C})$ are $\mathfrak{su}(1, 3)$, $\mathfrak{su}(2, 2)$, $\mathfrak{su}^*(4)$ and $\mathfrak{sl}(4, \mathbb{R})$.

Then, \mathcal{H} is isomorphic to one of the previous Lie algebras. Recall that \mathcal{H} contains a simple Lie subalgebra isomorphic to $\mathfrak{sl}(3, \mathbb{R})$. From here $2 = \text{rank}_\mathbb{R}(\mathfrak{sl}(3, \mathbb{R})) \leq \text{rank}_\mathbb{R}(\mathcal{H})$. By [4, Table V], we have $\text{rank}_\mathbb{R}(\mathfrak{su}(1, 3)) = \text{rank}_\mathbb{R}(\mathfrak{su}^*(4)) = 1$. Then, \mathcal{H} cannot be isomorphic to either $\mathfrak{su}(1, 3)$ or $\mathfrak{su}^*(4)$. On the other hand, page 519 of [4] shows that $\mathfrak{su}(2, 2) \simeq \mathfrak{so}(4, 2)$. So, if $\mathcal{H} \simeq \mathfrak{su}(2, 2)$ then $\mathfrak{sl}(3, \mathbb{R})$ is isomorphic to a Lie subalgebra of $\mathfrak{so}(4, 2)$. In this case $\mathfrak{sl}(3, \mathbb{R})$ would have a non-trivial representation on a 6-dimensional vector space that preserves a non-degenerate symmetric bilinear form of signature (4, 2). By Lemma 1.1 this cannot be possible. Thus, $\mathcal{H} \simeq \mathfrak{su}(2, 2)$ is not possible. Then $\mathcal{H} \simeq \mathfrak{sl}(4, \mathbb{R})$. \square

5. PROOF OF THE MAIN THEOREM

In this section we assume M is a connected analytic pseudo-Riemannian manifold with $\dim(M) = 14$ and finite volume. We also assume that $\text{SL}(3, \mathbb{R})$ acts isometric and analytically on M with a dense orbit, therefore the action is locally free, such that the normal bundle to the foliation (obtained by the action) $T\mathcal{F}^\perp$ is not integrable. We study the structure of the manifold M through the analysis of the different possibilities of \mathcal{H} obtained in Section 4. As in the previous section, we use the notation of Lemma 3.3.

Let $x \in \widetilde{M}$ be an element that satisfies Lemma 3.3 and Theorem 3.5. By Lemma 4.2 and Lemma 4.3 we can reduce the structure of \mathcal{H} to only 2 options: (1) $\mathcal{W}(x) \subseteq \text{rad}(\mathcal{H})$ is a subalgebra or (2) \mathcal{H} is a simple Lie algebra. In this section we analyze these cases and, assuming that $T\mathcal{F}^\perp$ is not integrable then, we will see that we can eliminate the first possibility.

5.1. $\mathcal{W}(x) \subseteq \text{rad}(\mathcal{H})$ is a subalgebra. Here, we assume that $\mathcal{W}(x)$ is a subalgebra of $\text{rad}(\mathcal{H})$ for some $x \in \widetilde{M}$.

Since $\text{rad}(\mathcal{H})$ is a solvable Lie algebra then, as case (c) in Lemma 4.2 and case (g) in 4.3, $\mathcal{W}(x) = \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is a 2-step nilpotent or an abelian Lie subalgebra.

By Lemma 3.3 and since $\mathcal{W}(x)$ is a subalgebra, we have that $\mathcal{G}(x) \oplus \mathcal{W}(x)$ is a Lie subalgebra of \mathcal{H} . Thus, $\mathcal{G}(x) \oplus \mathcal{W}(x)$ is isomorphic, as Lie algebra, to the semidirect product $\mathfrak{sl}(3, \mathbb{R}) \ltimes \mathfrak{w}$, where \mathfrak{w} is an $\mathfrak{sl}(3, \mathbb{R})$ -module, 2-step nilpotent or abelian Lie algebra isomorphic to $\mathcal{W}(x)$. Choose an isomorphism of Lie algebras $\psi : \mathfrak{sl}(3, \mathbb{R}) \ltimes \mathfrak{w} \rightarrow \mathcal{H}(x)$ that maps $\mathfrak{sl}(3, \mathbb{R})$ onto $\mathcal{G}(x)$ and \mathfrak{w} onto $\mathcal{W}(x)$.

Let $\widetilde{\text{SL}}(3, \mathbb{R}) \ltimes W$ be a simply connected Lie group such that $\text{Lie}(\widetilde{\text{SL}}(3, \mathbb{R}) \ltimes W) = \mathfrak{sl}(3, \mathbb{R}) \ltimes \mathfrak{w}$, where the group structure on W is induced considering the action of $\mathfrak{sl}(3, \mathbb{R})$ on \mathfrak{w} . By Lemma 2.8, there exists an analytic isometric right action of $\widetilde{\text{SL}}(3, \mathbb{R}) \ltimes W$ on \widetilde{M} such that $\psi(X) = X^*$ for every $X \in \mathfrak{sl}(3, \mathbb{R}) \ltimes \mathfrak{w}$.

Since \mathcal{H} centralizes the left $\widetilde{\text{SL}}(3, \mathbb{R})$ -action, then the right $(\widetilde{\text{SL}}(3, \mathbb{R}) \ltimes W)$ -action centralizes the left $\widetilde{\text{SL}}(3, \mathbb{R})$ -action as well and preserves both $T\mathcal{F}$ and $T\mathcal{F}^\perp$.

Using the right $(\widetilde{\mathrm{SL}}(3, \mathbb{R}) \ltimes \mathbb{W})$ -action on \widetilde{M} , we consider the following map:

$$(5.1) \quad p : \widetilde{\mathrm{SL}}(3, \mathbb{R}) \ltimes \mathbb{W} \rightarrow \widetilde{M}, \quad h \mapsto x \cdot h,$$

for $h \in \widetilde{\mathrm{SL}}(3, \mathbb{R}) \ltimes \mathbb{W}$. This action is $(\widetilde{\mathrm{SL}}(3, \mathbb{R}) \ltimes \mathbb{W})$ -equivariant by the right action on its domain. If e and 0 denote the identity element in the subgroups $\widetilde{\mathrm{SL}}(3, \mathbb{R})$ and \mathbb{W} , respectively, then

$$(5.2) \quad \begin{aligned} dp_{(e,0)} : \mathfrak{sl}(3, \mathbb{R}) \ltimes \mathfrak{w} &\rightarrow \mathcal{G}(x) \oplus \mathcal{W}(x) \rightarrow T_x \widetilde{M} \\ X &\mapsto X^* \mapsto X_x^*. \end{aligned}$$

Since $\psi(X) = X^*$ for all $X \in \mathfrak{sl}(3, \mathbb{R}) \ltimes \mathfrak{w}$, by Lemma 3.3, $dp_{(e,0)}$ maps $\mathfrak{sl}(3, \mathbb{R})$ onto $T_x \mathcal{F}$ and \mathfrak{w} onto $T_x \mathcal{F}^\perp$. Therefore, p is a local diffeomorphism at $(e, 0)$.

For every $w \in \mathbb{W}$, let R_w denote the map on $\widetilde{\mathrm{SL}}(3, \mathbb{R}) \ltimes \mathbb{W}$ and on \widetilde{M} given by the correspondence $y \mapsto y \cdot (e, w)$. Since \mathbb{W} is a subgroup of $\widetilde{\mathrm{SL}}(3, \mathbb{R}) \ltimes \mathbb{W}$ we have that $R_w(\mathbb{W}) = \mathbb{W}$.

Let $P = p(e \times \mathbb{W})$, which defines a submanifold of \widetilde{M} in a neighborhood of $x = p(e, 0)$. Here, by the previous remarks, we have that

$$T_{p(e,0)} P = dp_{(e,0)}(T_{(e,0)}(e \times \mathbb{W})) = T_{p(e,0)} \mathcal{F}^\perp,$$

which with the equivariance of p implies that

$$\begin{aligned} T_{p(e,w)} P &= dp_{(e,w)}(T_{(e,w)}(e \times \mathbb{W})) = dp_{(e,w)}(d(R_w)_{(e,0)}(T_{(e,0)}(e \times \mathbb{W}))) \\ &= d(R_w \circ p)_{(e,0)}(T_{(e,0)}(e \times \mathbb{W})) = d(R_w)_{p(e,0)}(T_{p(e,0)} P) \\ &= d(R_w)_{p(e,0)} T_{p(e,0)} \mathcal{F}^\perp = T_{R_w(p(e,0))} \mathcal{F}^\perp = T_{p(e,w)} \mathcal{F}^\perp, \end{aligned}$$

we have used in the previous identities that R_w preserves the bundle $T\mathcal{F}^\perp$. This proves that P is an integral submanifold of $T\mathcal{F}^\perp$ passing through the element $x = p(e, 0)$.

By the left $\widetilde{\mathrm{SL}}(3, \mathbb{R})$ -action on \widetilde{M} we obtain by restriction to P the following map:

$$\phi : \widetilde{\mathrm{SL}}(3, \mathbb{R}) \times P \rightarrow \widetilde{M}, \quad (g, y) \mapsto g \cdot y,$$

whose differential at (e, x) is given by: $X + v \mapsto X_x^* + v$, with $X \in \mathfrak{sl}(3, \mathbb{R})$ and $v \in T_x P$. This shows that the differential at (e, x) is an isomorphism and therefore the map ϕ is a diffeomorphism from a neighborhood of (e, x) onto a neighborhood of x . Since the left $\widetilde{\mathrm{SL}}(3, \mathbb{R})$ -action preserves both $T\mathcal{F}$ and $T\mathcal{F}^\perp$, there is an integral submanifold of $T\mathcal{F}^\perp$ passing through every point in a neighborhood of x in \widetilde{M} . Thus, the tensor Θ considered in Lemma 2.3 vanishes in a neighborhood of x . Since all of our objects are analytic, this implies that Θ vanishes everywhere therefore Lemma 2.3 implies the integrability of $T\mathcal{F}^\perp$ everywhere in \widetilde{M} .

This last conclusion contradicts the assumption about the integrability of $T\mathcal{F}^\perp$.

5.2. \mathcal{H} is a simple Lie algebra. Here, we assume $\mathcal{H}(x)$ is a simple Lie algebra. By Lemma 4.2 and Lemma 4.3 we have two possibilities: (a) $\mathcal{H}_0(x) = 0$ and $\mathcal{H}(x) \simeq \mathfrak{g}_{2(2)}$ or (b) $\mathcal{H}_0(x) \simeq \mathbb{R}$ and $\mathcal{H}(x) \simeq \mathfrak{sl}(4, \mathbb{R})$

5.2.1. $\mathcal{H} \simeq \mathfrak{g}_{2(2)}$. If $\mathcal{H}_0(x) = 0$ and \mathcal{H} is a simple Lie algebra then we have proved that $\mathcal{H} \simeq \mathfrak{g}_{2(2)}$, therefore our the following result:

Lemma 5.1. *There is an isomorphism*

$$\psi : \mathfrak{g}_{2(2)} = \mathfrak{sl}(3, \mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*} \rightarrow \mathcal{G}(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^* = \mathcal{H}$$

of Lie algebras. In particular, we have that ψ is an isomorphism of $\mathfrak{sl}(3, \mathbb{R})$ -modules which preserves the summands in that order.

Proof. Let $\psi : \mathfrak{g}_{2(2)} \rightarrow \mathcal{H}$ be an isomorphism of simple Lie algebras. Thus, $\psi^{-1}(\mathcal{G}(x))$ is a Lie subalgebra which provides a direct sum decomposition of $\mathfrak{g}_{2(2)}$ into irreducible $\mathfrak{sl}(3, \mathbb{R})$ -modules. Such decomposition, by [3, Chapter 22], is given by $\psi^{-1}(\mathcal{G}(x)) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*}$. Therefore,

$$\psi(\psi^{-1}(\mathcal{G}(x))) \oplus \psi(\mathbb{R}^3) \oplus \psi(\mathbb{R}^{3*}) = \mathcal{G}(x) \oplus \psi(\mathbb{R}^3) \oplus \psi(\mathbb{R}^{3*}),$$

is a decomposition of \mathcal{H} into irreducible $\mathfrak{sl}(3, \mathbb{R})$ -modules.

Recall that we have a previous decomposition of \mathcal{H} into irreducible $\mathfrak{sl}(3, \mathbb{R})$ -modules

$$\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x).$$

Comparing the two decomposition of \mathcal{H} into irreducible $\mathfrak{sl}(3, \mathbb{R})$ -modules we obtain our desired result. \square

We fix an isomorphism of Lie algebras $\psi : \mathfrak{g}_{2(2)} \rightarrow \mathcal{H}$ as in the previous lemma. Let $G_{2(2)}$ denote a simply connected Lie group such that $\text{Lie}(G_{2(2)}) = \mathfrak{g}_{2(2)}$. By Lemma 2.8, there exists an analytic isometric right $G_{2(2)}$ -action on \widetilde{M} such that $\psi(X) = X^*$, for all $X \in \mathfrak{g}_{2(2)}$. Now, we consider the map:

$$\begin{aligned} p : G_{2(2)} &\rightarrow \widetilde{M} \\ g &\mapsto x \cdot g, \end{aligned}$$

that satisfies $dp_e(X) = X_x^* = \psi(X)$ for every $X \in \mathfrak{g}_{2(2)}$. Thus, by our choice of ψ and Lemma 3.3, dp_e is an isomorphism that maps $\mathfrak{sl}(3, \mathbb{R})$ on $T_x\mathcal{F}$ and $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$ onto $T_x\mathcal{F}^\perp$. Since p is $G_{2(2)}$ -equivariant for the right action on its domain, then we have a local diffeomorphism.

Lemma 5.2. *Let \bar{g} be the metric on $\mathfrak{g}_{2(2)}$ defined as the pullback under dp_e of the metric g_x on $T_x\widetilde{M}$ then \bar{g} is $\mathfrak{sl}(3, \mathbb{R})$ -invariant.*

Proof. By properties of dp_e and the isomorphism ψ , we need only to prove that the metric on \mathcal{H} defined as the pullback of g_x by the evaluation map, $X \mapsto X_x$, is $\mathcal{G}(x)$ -equivariant.

Let \tilde{g} be the metric on \mathcal{H} obtained of this way. Let $X, Y, Z \in \mathcal{H}$ be given with $X \in \mathcal{G}(x)$. By Lemma 2.7 there exists $X_0 \in \mathfrak{sl}(3, \mathbb{R})$ such that $X = \rho_x(X_0) + X_0^*$, where ρ_x is the homomorphism in Proposition 2.2 and X_0^* is the vector field on \widetilde{M} induced by X_0 through the left $\widetilde{\text{SL}}(3, \mathbb{R})$ -action. Therefore

$$\begin{aligned} \tilde{g}([X, Y], Z) &= g_x([X, Y]_x, Z_x) = g([X, Y], Z)|_x = g([\rho_x(X_0) + X_0^*, Y], Z)|_x \\ &= g([\rho_x(X_0), Y], Z)|_x = \rho_x(X_0)g(Y, Z)|_x - g(Y, [\rho_x(X_0), Z])|_x \\ &= -g(Y, [\rho_x(X_0), Z])|_x = -g(Y, [\rho_x(X_0) + X_0^*, Z])|_x \\ &= -g(Y, [X, Z])|_x = -g_x(Y_x, [X_x, Z_x]) = -\tilde{g}(Y, [X, Z]). \end{aligned}$$

Where we have used the fact that \mathcal{H} centralizes X_0^* and that $\rho_x(X_0)$ is a Killing vector field, for the metric g , which vanishes in x . Thus, we take \tilde{g} as the pullback of \tilde{g} by the isomorphism ψ to obtain the desired result. \square

Now, by the previous lemma and Lemma A.4, we can rescale the metric along the bundles $T\mathcal{F}$ and $T\mathcal{F}^\perp$ in \tilde{M} such that the new metric, \hat{g} , on \tilde{M} satisfies that $K = (dp_e)^*(\hat{g}_x)$, is the Killing form on $\mathfrak{g}_{2(2)}$.

Since the elements of $\mathcal{H} \subset \mathrm{Kill}(\tilde{M})$ preserve the direct sum decomposition, $T\tilde{M} = T\mathcal{F} \oplus T\mathcal{F}^\perp$, then $\mathcal{H} \subset \mathrm{Kill}(\tilde{M}, \hat{g})$. Note that \hat{g} is invariant under both the left $\widetilde{\mathrm{SL}}(3, \mathbb{R})$ -action and the right $G_{2(2)}$ -action on \tilde{M} . Observe, also, that the metric \hat{g} can be obtained from the lift of a correspondingly rescaled metric on M .

Consider the bi-invariant metric on $G_{2(2)}$ induced by the Killing form K , which we denote with the same letter. The previous argument and discussion imply that the local diffeomorphism

$$p : (G_{2(2)}, K) \rightarrow (\tilde{M}, \hat{g})$$

is a local isometry. This last property of p , the completeness of $(G_{2(2)}, K)$ and the simply completeness of (\tilde{M}, \hat{g}) imply, by Corollary 20 in [5, p. 202], that p is an isometry.

Proposition 5.3. *Let M be an analytic connected finite volume pseudo-Riemannian manifold with $\dim(M) = 14$. If M is complete, admits an analytic and isometric right $\mathrm{SL}(3, \mathbb{R})$ -action with a dense orbit such that \mathcal{H} (Lemma 2.6) is a simple Lie algebra with $\dim(\mathcal{H}) = 14$. Then there exists an analytic diffeomorphism $p : G_{2(2)} \rightarrow \tilde{M}$ and an analytic isometric right $G_{2(2)}$ -action on \tilde{M} such that:*

- (i) *On \tilde{M} , the left $\widetilde{\mathrm{SL}}(3, \mathbb{R})$ -action and the right $G_{2(2)}$ -action commute,*
- (ii) *p is $G_{2(2)}$ -equivariant for the right $G_{2(2)}$ -action on its domain,*
- (iii) *for a pseudo-Riemannian metric \hat{g} in \tilde{M} obtained by rescaling the original metric on the summands of the decomposition $T\tilde{M} = T\mathcal{F} \oplus T\mathcal{F}^\perp$, the map*

$$p : (G_{2(2)}, K) \rightarrow (\tilde{M}, \hat{g})$$

is an isometry where K is the bi-invariant metric on $G_{2(2)}$ induced from the Killing form of its Lie algebra.

Considering $G_{2(2)}$ with the bi-invariant pseudo-Riemannian metric K , induced by the Killing form of its Lie algebra, we can assume that $(G_{2(2)}, K)$ is the isometric universal covering space of (\tilde{M}, \hat{g}) .

By Proposition 4.5 of [7] we have that the isometry group of the pseudo-Riemannian manifold $(G_{2(2)}, K)$, which we denote by $\mathrm{Iso}(G_{2(2)})$, has only a finite number of connected components. Such proposition also shows that

$$\mathrm{Iso}(G_{2(2)})_0 = L(G_{2(2)})R(G_{2(2)}),$$

where $L(G_{2(2)})$ and $R(G_{2(2)})$ are the subgroups of left and right translations on $G_{2(2)}$, respectively.

Let $\varrho : \widetilde{\mathrm{SL}}(3, \mathbb{R}) \rightarrow \mathrm{Iso}(G_{2(2)})$ be the homomorphism induced by the isometric left $\widetilde{\mathrm{SL}}(3, \mathbb{R})$ -action on $G_{2(2)}$. From the previous observations, the covering

$$G_{2(2)} \times G_{2(2)} \rightarrow L(G_{2(2)})R(G_{2(2)})$$

yields the existence of homomorphisms $\varrho_1, \varrho_2 : \widetilde{\mathrm{SL}}(3, \mathbb{R}) \rightarrow G_{2(2)}$ such that

$$\varrho(g) = L_{\varrho_1(g)} \circ R_{\varrho_2(g)^{-1}} \quad \forall g \in \widetilde{\mathrm{SL}}(3, \mathbb{R}).$$

By Proposition 5.3, we have that $\varrho(g) \circ R_h = R_h \circ \varrho(g)$ for all $g \in \widetilde{\mathrm{SL}}(3, \mathbb{R})$ and $h \in G_{2(2)}$, which implies that $\varrho_2(\widetilde{\mathrm{SL}}(3, \mathbb{R}))$ is contained in the center of $G_{2(2)}$, thence $\varrho(g) = L_{\varrho_1(g)}$ for all $g \in \widetilde{\mathrm{SL}}(3, \mathbb{R})$. Thus, the $\widetilde{\mathrm{SL}}(3, \mathbb{R})$ -action on $G_{2(2)}$ is induced by the homomorphism $\varrho_1 : \widetilde{\mathrm{SL}}(3, \mathbb{R}) \rightarrow G_{2(2)}$ and the left action of $G_{2(2)}$ onto itself. Note, by our hypotheses, that the homomorphism ϱ_1 is non-trivial.

By Proposition 5.3 we have that $\pi_1(M) \subset \mathrm{Iso}(G_{2(2)})$ and by the previous observations $\Gamma_1 = \pi_1(M) \cap \mathrm{Iso}_0(G_{2(2)})$ is a finite index subgroup of $\pi_1(M)$. Therefore, for every $\gamma \in \Gamma_1$ there exist $h_1, h_2 \in G_{2(2)}$ such that $\gamma = L_{h_1} \circ R_{h_2}$.

Since the left $\widetilde{\mathrm{SL}}(3, \mathbb{R})$ -action on $G_{2(2)}$ is the lift of an action on M , this left $\widetilde{\mathrm{SL}}(3, \mathbb{R})$ -action commutes with the Γ_1 -action. Applying that property to $L_{h_1} \circ R_{h_2} = \gamma \in \Gamma_1$ we obtain that $L_{h_1} \circ L_{\varrho_1(g)} = L_{\varrho_1(g)} \circ L_{h_1}$ for all $g \in \widetilde{\mathrm{SL}}(3, \mathbb{R})$, thus $\Gamma_1 \in L(Z(\widetilde{\mathrm{SL}}(3, \mathbb{R})))R(G_{2(2)})$ where $Z(\widetilde{\mathrm{SL}}(3, \mathbb{R}))$ is the centralizer of $\varrho_1(\widetilde{\mathrm{SL}}(3, \mathbb{R}))$ in $G_{2(2)}$. By Lemma A.2, the center of $G_{2(2)}$ has finite index in $Z(\widetilde{\mathrm{SL}}(3, \mathbb{R}))$ and therefore $R(G_{2(2)})$ has finite index in $L(\widetilde{\mathrm{SL}}(3, \mathbb{R}))R(G_{2(2)})$. In particular, $\Gamma = \Gamma_1 \cap R(G_{2(2)})$ is a finite index subgroup of Γ_1 and also of $\pi_1(M)$.

The natural identification of $R(G_{2(2)})$ with $G_{2(2)}$ realizes Γ as a discrete subgroup of $G_{2(2)}$ such that $G_{2(2)}/\Gamma$ is a finite covering space of M .

Let $\varphi : G_{2(2)}/\Gamma \rightarrow M$ be the corresponding covering map. For the left $\widetilde{\mathrm{SL}}(3, \mathbb{R})$ -action on $G_{2(2)}/\Gamma$ given by the homomorphism $\varrho_1 : \widetilde{\mathrm{SL}}(3, \mathbb{R}) \rightarrow G_{2(2)}$, the constructions in the previous paragraphs show that the map φ is $\widetilde{\mathrm{SL}}(3, \mathbb{R})$ -equivariant. Finally, we note that φ is a local isometry for the metric \widehat{g} , on \widetilde{M} considered in Proposition 5.3.

Next, we are going to show that the subgroup Γ is a lattice in $G_{2(2)}$. For the proof of that result it is enough to prove that M has finite volume in the metric \widehat{g} . Recall, we are assuming that M has finite volume in its original metric.

Lemma 5.4. *If vol and $\mathrm{vol}_{\widehat{g}}$ denote the volume elements on M for the original metric and the rescaled metric, respectively. Then, there is some constant $C > 0$ such that $\mathrm{vol}_{\widehat{g}} = C \mathrm{vol}$.*

Proof. We consider $(x^1, x^2, \dots, x^{14})$ some coordinate of M in a neighborhood U of a given point such that (x^1, \dots, x^8) defines a set of coordinates of the leaves of the foliation \mathcal{F} in such neighborhood. For the original metric g on M , consider the orthogonal bundle $T\mathcal{F}^\perp$ and a set of 1-forms $\theta^1, \dots, \theta^6$ that define a basis for its dual $(T\mathcal{F}^\perp)^*$ at every point in U . Thus, in U the metric g has an expression of the form:

$$g = \sum_{i,j=1}^8 l_{ij} dx^i \otimes dx^j + \sum_{i,j=1}^6 h_{ij} \theta^i \otimes \theta^j.$$

From this and the definition of the volume element as an 14-form, we have:

$$\mathrm{vol} = \sqrt{|\det(l_{ij}) \det(h_{ij})|} dx^1 \wedge \dots \wedge dx^8 \wedge \theta^1 \wedge \dots \wedge \theta^6.$$

On the other hand, since the metric \widehat{g} is obtained by rescaling g along the bundles $T\mathcal{O}$ and $T\mathcal{O}^\perp$, then has an expression of the form:

$$\widehat{g} = \sum_{i,j=1}^8 C_1 l_{ij} dx^i \otimes dx^j + \sum_{i,j=1}^6 C_2 h_{ij} \theta^i \otimes \theta^j.$$

for some constants $C_1, C_2 \neq 0$. Therefore, the volume element of \widehat{g} satisfies:

$$\begin{aligned} \mathrm{vol}_{\widehat{g}} &= \sqrt{|\det(C_1 l_{ij}) \det(C_2 h_{ij})|} dx^1 \wedge \dots \wedge dx^8 \wedge \theta^1 \wedge \dots \wedge \theta^6 \\ &= \sqrt{|C_1^8 C_2^6|} \mathrm{vol}. \end{aligned}$$

□

5.2.2. $\mathcal{H} \simeq \mathfrak{sl}(4, \mathbb{R})$. Here, we assume $\mathcal{H}_0(x) \simeq \mathbb{R}$ and that \mathcal{H} is a simple Lie algebra then, by Lemma 4.3(2), our centralizer \mathcal{H} is isomorphic to $\mathfrak{sl}(4, \mathbb{R})$. Therefore, we have our following result

Lemma 5.5. *There is an isomorphism*

$$\psi : \mathfrak{sl}(4, \mathbb{R}) = \mathfrak{sl}(3, \mathbb{R}) \oplus \mathbb{R} \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*} \rightarrow \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x) = \mathcal{H}$$

of Lie algebras that preserves the summands in that order. In particular, ψ is an isomorphism of $\mathfrak{sl}(3, \mathbb{R})$ -modules.

Proof. Similar to the proof of Lemma 5.1. □

Let us fix an isomorphism of Lie algebras $\psi : \mathfrak{sl}(4, \mathbb{R}) \rightarrow \mathcal{H}$ as in the previous lemma. Let $\widetilde{\mathrm{SL}}(4, \mathbb{R})$ denote the universal covering of $\mathrm{SL}(4, \mathbb{R})$ then $\mathrm{Lie}(\widetilde{\mathrm{SL}}(4, \mathbb{R})) = \mathfrak{sl}(4, \mathbb{R})$. By Lemma 2.8, there exists an analytic isometric right $\widetilde{\mathrm{SL}}(4, \mathbb{R})$ -action on \widetilde{M} such that $\psi(X) = X^*$, for every $X \in \mathfrak{sl}(4, \mathbb{R})$. This right action centralizes the left $\widetilde{\mathrm{SL}}(3, \mathbb{R})$ -action on \widetilde{M} and thus preserves the bundles $T\mathcal{F}$ and $T\mathcal{F}^\perp$.

Given the previous right $\widetilde{\mathrm{SL}}(4, \mathbb{R})$ -action on \widetilde{M} we define the map $p : \widetilde{\mathrm{SL}}(4, \mathbb{R}) \rightarrow \widetilde{M}$ defined as $g \mapsto x \cdot g$, for all $g \in \widetilde{\mathrm{SL}}(4, \mathbb{R})$. Observe that this map is $\widetilde{\mathrm{SL}}(4, \mathbb{R})$ -invariant for the right action on its domain satisfying $dp_e(X) = X_x^* = \psi(X)$ for every $X \in \mathfrak{sl}(3, \mathbb{R})$. Note that dp_e is surjective with $\ker(dp_e) = \psi^{-1}(\mathcal{H}_0(x))$.

Let H be a connected subgroup of $\widetilde{\mathrm{SL}}(4, \mathbb{R})$ such that $\mathrm{Lie}(H) = \psi^{-1}(\mathcal{H}_0(x))$. Here, H is not a compact subgroup and, by exercise (vi) in [4, p. 152], a closed subgroup. Hence, the map

$$\overline{p} : H \backslash \widetilde{\mathrm{SL}}(4, \mathbb{R}) \rightarrow \widetilde{M}, \quad [g] = Hg \mapsto x \cdot g,$$

for all $Hg \in H \backslash \widetilde{\mathrm{SL}}(4, \mathbb{R}) \rightarrow \widetilde{M}$, is well defined. Observe that $T_{He}(H \backslash \widetilde{\mathrm{SL}}(4, \mathbb{R})) = \mathfrak{sl}(3, \mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*}$.

By our choice of the map ψ we have that $d\overline{p}_{He}$ is an isomorphism which maps $\mathfrak{sl}(3, \mathbb{R})$ onto $T\mathcal{F}$ and $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$ onto $T\mathcal{F}^\perp$. Since \overline{p} is an $\widetilde{\mathrm{SL}}(4, \mathbb{R})$ -equivariant map for the right action on its domain then \overline{p} is an analytic local diffeomorphism at He .

The definition of the map \overline{p} and Lemma 5.5 imply that that $d\overline{p}_{He} = ev_x \circ \psi$ restricted to the subspace $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*}$. This map induces a metric on $H \backslash \widetilde{\mathrm{SL}}(4, \mathbb{R})$, which is the result of the next lemma.

Lemma 5.6. *Let \bar{g} be the metric on $T_{He}(H \backslash \widetilde{\mathrm{SL}}(4, \mathbb{R}))$ defined as the pullback under $d\overline{p}_{[e]}$ of the metric g_x on $T_x \widetilde{M}$, then \bar{g} is $\mathfrak{sl}(3, \mathbb{R})$ -invariant.*

Proof. Since $d\bar{p}_{[e]} = ev_x \circ \psi|_{(\mathfrak{sl}(3, \mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*})}$, then $d\bar{p}_{He}$ is a homomorphism of $\mathfrak{sl}(3, \mathbb{R})$ -modules. Recall that the $\mathfrak{sl}(3, \mathbb{R})$ -module structure in $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*}$ is given by the subalgebra $\mathfrak{sl}(3, \mathbb{R})$ and in $T_x \widetilde{M}$ by the subalgebra $\rho_x(\mathfrak{sl}(3, \mathbb{R})(x))$, as in Proposition 2.2(4).

Since the metric g in $T_x \widetilde{M}$ is invariant under the action of $\rho_x(\mathfrak{sl}(3, \mathbb{R})(x))$ we have that if $u, v \in T_{He}(H \setminus \widetilde{SL}(4, \mathbb{R}))$ and $X \in \mathfrak{sl}(3, \mathbb{R})$ then, by Lemma 2.7 there are $U, V \in \mathcal{H}$ such that $U_x = u$ and $V_x = v$ and by Proposition 2.2 $\hat{\rho}_x(X) \in \mathcal{G}(x)$ and $X_0 = \psi^{-1}(\hat{\rho}_x(X)) \in T_{He}(H \setminus \widetilde{SL}(4, \mathbb{R}))$ satisfying that

$$\begin{aligned} \bar{g}([X, v], w) &:= \bar{g}([X_0, v], w) = g_x(d\bar{p}_{He}([X_0, v]), d\bar{p}_{He}(w)) \\ &= g_x((ev_x \circ \psi)[X_0, v], (ev_x \circ \psi)(w)) = g_x(\psi([X_0, v])_x, \psi(w)_x) \\ &= g(\psi([X_0, v]), \psi(w))|_x = g([\psi(X_0), \psi(v)], \psi(w))|_x \\ &= g([\hat{\rho}_x(X), V], W)|_x = g([\rho_x(X) + X^*, V], W)|_x \\ &= g([\rho_x(X), V], W)|_x = \rho_x(X)(g(V, W))|_x - g(V, [\rho_x(X), W])|_x \\ &= -g(V, [\rho_x(X), W])|_x = -g(V, [\rho_x(X) + X^*, W])|_x \\ &= -g(V, [\hat{\rho}_x(X), W])|_x = -g(\psi(v), [\psi(X_0), \psi(w)])|_x \\ &= -g(\psi(v), \psi([X_0, w]))|_x = -g_x(df_{[e]}(v), df_{He}([X_0, w])) \\ &= -\bar{g}(v, [X_0, w]) = -\bar{g}(v, [X, w]), \end{aligned}$$

Recall that \mathcal{H} centralizes X^* and $\rho_{x_0}(X)$ is a Killing field for the metric g in \widetilde{M} . Therefore, by properties of the maps ev_{x_0} and φ , the metric \bar{g} in

$$T_{He}(H \setminus \widetilde{SL}(4, \mathbb{R})) = \mathfrak{sl}(3, \mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*}.$$

is $\mathfrak{sl}(3, \mathbb{R})$ -invariant. \square

Next, we analyze the pseudo-Riemannian metric structure of the analytic manifold $H \setminus \widetilde{SL}(4, \mathbb{R})$.

First, let K_n be the Killing form on $\mathfrak{sl}(n, \mathbb{R})$ with $n \geq 2$. Recall that $K_n(X, Y) = 2n \cdot \text{tr}(XY)$ for all $X, Y \in \mathfrak{sl}(n, \mathbb{R})$. By the decomposition of $\mathfrak{sl}(4, \mathbb{R})$ as $\mathfrak{sl}(3, \mathbb{R})$ -module in Section 1 and the definition of the Killing form we have that $\mathfrak{sl}(3, \mathbb{R})$, $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$ and $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*}$ (as subspaces of $\mathfrak{sl}(4, \mathbb{R})$) are non-degenerate with respect to the Killing form K_4 .

Denote by K^1, K^2 and K the restriction of K_4 to the subspaces $\mathfrak{sl}(3, \mathbb{R})$, $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$ and $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*}$, respectively. Since K_4 is invariant by the adjoint action of $\mathfrak{sl}(4, \mathbb{R})$ then it is clear that K_4 is also invariant under the adjoint action of $\mathfrak{sl}(3, \mathbb{R})$, by its inclusion in $\mathfrak{sl}(4, \mathbb{R})$. Therefore, K^1, K^2 and K are invariant by the adjoint action of $\mathfrak{sl}(3, \mathbb{R})$.

Remark 5.7. By the definition and properties of K^1 and since $\mathfrak{sl}(3, \mathbb{R})$ is a simple Lie algebra we have, by Schur's Lemma, that $K^1 = c_1 K_3$, with $c_1 \neq 0$. On the other hand, by Section 1 and Lemma A.3, we have that $K^2 = K_4|_{\mathbb{R}^3 \oplus \mathbb{R}^{3*}}$ is a non-zero multiple of a unique $\mathfrak{sl}(3, \mathbb{R})$ -invariant symmetric bilinear form on $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$.

Let $\pi : \widetilde{SL}(4, \mathbb{R}) \rightarrow H \setminus \widetilde{SL}(4, \mathbb{R})$ be the natural quotient map. With respect to the previous map, we assume that

$$(5.3) \quad d\pi_0 = d\pi|_{\mathfrak{sl}(3, \mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*}} : \mathfrak{sl}(3, \mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*} \rightarrow T_{He} H \setminus \widetilde{SL}(4, \mathbb{R})$$

is a linear isometry.

By its construction, we have that the quotient space $H \backslash \widetilde{\mathrm{SL}}(4, \mathbb{R})$ is a *reductive coset* (see [5, p. 310]). On the other hand, by properties of the Killing form K_4 and the definition of K , we have that K is $\mathrm{Ad}(H)$ -invariant. Thus, such result and the isometry (5.3) imply, by Proposition 22 in [5, p. 311], the $\widetilde{\mathrm{SL}}(4, \mathbb{R})$ -invariance on $H \backslash \widetilde{\mathrm{SL}}(4, \mathbb{R})$. Therefore, by Lemma 5.5, the properties of K_4 and the isometry (5.3) we have that the manifold $H \backslash \widetilde{\mathrm{SL}}(4, \mathbb{R})$ is a *naturally reductive homogeneous space*, (see [5, p. 312]).

As a consequence that our manifold is a naturally reductive homogeneous space we have that $H \backslash \widetilde{\mathrm{SL}}(4, \mathbb{R})$ is complete (see [5, p. 313]), hence, by Lemma 24 in [5, p. 312], our quotient map $\pi : \widetilde{\mathrm{SL}}(4, \mathbb{R}) \rightarrow H \backslash \widetilde{\mathrm{SL}}(4, \mathbb{R})$ is a pseudo-Riemannian submersion.

Next, we show that we can rescale the metric on \widetilde{M} such that the pullback of this new metric, with respect to the map \bar{p} , implies that (5.3) is, effectively, a linear isometry. But, first we need the following result.

Lemma 5.8. *Let $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ be the inner products on $\mathfrak{sl}(3, \mathbb{R})$ and $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$, respectively. Assume that $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ are $\mathfrak{sl}(3, \mathbb{R})$ -invariant. Then there exist $c_1, c_2 \in \mathbb{R}$ such that*

$$c_1 \langle \cdot, \cdot \rangle_1 + c_2 \langle \cdot, \cdot \rangle_2,$$

is K , the Killing form of $\mathfrak{sl}(4, \mathbb{R})$ restricted to $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3}$.*

Proof. Recall, Schur's Lemma implies that in \mathfrak{g} , a simple real Lie algebra with a simple complexification, any \mathfrak{g} -invariant non-degenerate symmetric bilinear form on \mathfrak{g} is a multiple by a real scalar of the Killing form.

On the other hand, we have proved in Lemma A.4 that there is, up to a multiple by a real scalar, a unique $\mathfrak{sl}(3, \mathbb{R})$ -invariant non-degenerate bilinear form on $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$.

Now, the result follows from previous results. \square

Remark 5.9. By the results in Lemmas 5.6 and 5.8, we can rescale the metric g along the bundles $T\mathcal{F}$ and $T\mathcal{F}^\perp$ on \widetilde{M} such that the new metric, \widehat{g} , satisfies $(d\bar{f}_{He})^*(\widehat{g}_x) = K$, the Killing form on $\mathfrak{sl}(4, \mathbb{R})$ restricted to $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*}$.

Since the elements in \mathcal{H} preserve the decomposition $T\widetilde{M} = T\mathcal{F} \oplus T\mathcal{F}^\perp$, then $\mathcal{H} \subset \mathrm{Kill}(\widetilde{M}, \widehat{g})$. That is, the elements in \mathcal{H} are Killing vector fields for the metric \widehat{g} rescaled as in Remark 5.9. Therefore, \widehat{g} is invariant under the right $\widetilde{\mathrm{SL}}(4, \mathbb{R})$ -action. In a similar way, the left $\widetilde{\mathrm{SL}}(3, \mathbb{R})$ -action on \widetilde{M} preserves the rescaled metric \widehat{g} . Note that \widehat{g} on \widetilde{M} can be obtained as the lift of a correspondingly rescaled metric \widehat{g} in M .

Remark 5.9 implies that the local diffeomorphism

$$\bar{p} : (H \backslash \widetilde{\mathrm{SL}}(4, \mathbb{R}), K) \rightarrow (\widetilde{M}, \widehat{g})$$

is a local isometry. Thus, the completeness of $(H \backslash \widetilde{\mathrm{SL}}(4, \mathbb{R}), K)$ and the simple completeness of \widetilde{M} imply, by Corollary 29 in [5, p. 202], that \bar{p} is an isometry. Therefore our next result.

Proposition 5.10. *Let M be a connected analytic pseudo-Riemannian manifold. Suppose that M is complete, has finite volume and admits an analytic and isometric $\mathrm{SL}(3, \mathbb{R})$ -action with a dense orbit such that the centralizer of this action, \mathcal{H} , is a Lie algebra simple with $\dim(\mathcal{H}) = 15$. If $\dim(M) = 14$, then there exists an analytic*

diffeomorphism $\bar{p} : H \backslash \widetilde{\mathrm{SL}}(4, \mathbb{R}) \rightarrow \widetilde{M}$ and an analytic isometric right $\widetilde{\mathrm{SL}}(4, \mathbb{R})$ -action on \widetilde{M} such that:

- (1) on \widetilde{M} the left $\widetilde{\mathrm{SL}}(3, \mathbb{R})$ -action and the right $\widetilde{\mathrm{SL}}(4, \mathbb{R})$ -action commute with each other,
- (2) \bar{p} is $\widetilde{\mathrm{SL}}(4, \mathbb{R})$ -equivariant for the right $\widetilde{\mathrm{SL}}(4, \mathbb{R})$ -action on its domain,
- (3) for a pseudo-Riemannian metric \widehat{g} in \widetilde{M} obtained by rescaling the original metric on the summands of $T\widetilde{M} = T\mathcal{F} \oplus T\mathcal{F}^\perp$, the map

$$\bar{p} : (H \backslash \widetilde{\mathrm{SL}}(4, \mathbb{R}), K) \rightarrow (\widetilde{M}, \widehat{g})$$

is an isometry where K is the metric on $H \backslash \widetilde{\mathrm{SL}}(4, \mathbb{R})$ which makes of the quotient map π a pseudo-Riemannian submersion.

APPENDIX A.

With the definitions and results from [6] we have the next Lemma.

Lemma A.1. *Suppose that $\rho : \mathfrak{sl}(3, \mathbb{R}) \rightarrow \mathfrak{g}_{2(2)}$ is an injective Lie algebra homomorphism. Then $\mathfrak{s} = \rho(\mathfrak{sl}(3, \mathbb{R}))$ is a subalgebra of $\mathfrak{g}_{2(2)}$ with its centralizer, $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$, equal to zero.*

Lemma A.2. *Suppose that G is a connected Lie group locally isomorphic to $G_{2(2)}$ and consider $\iota : \widetilde{\mathrm{SL}}(3, \mathbb{R}) \rightarrow G$ a non trivial homomorphism of Lie groups. Then, the centralizer $Z_G(\iota(\widetilde{\mathrm{SL}}(3, \mathbb{R})))$ of $\iota(\widetilde{\mathrm{SL}}(3, \mathbb{R}))$ in G contains $Z(G)$ (the center of G) as a finite index subgroup.*

Proof. Let $S = \iota(\widetilde{\mathrm{SL}}(3, \mathbb{R}))$ and denote the Lie algebra of S as \mathfrak{s} . Since $Z(G) \subseteq Z_G(S)$ and, by the previous Lemma, $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) = 0$ then $Z_G(S)$ and $Z(G)$ are discrete. The proof that $Z_G(S)$ is finite is a consequence of Lemma 1.1.3.7 in [12]. \square

We prove now the relationship between the Killing form on the simple Lie group $\mathfrak{g}_{2(2)}$ and the $\mathfrak{sl}(3, \mathbb{R})$ -invariant bilinear form, both obtained in $\mathfrak{sl}(3, \mathbb{R})$ and $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$. But before we need the following result.

Lemma A.3. *There is, up to a multiple by a real scalar, exactly one $\mathfrak{sl}(3, \mathbb{R})$ -invariant non-degenerate bilinear form on $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$.*

Proof. We have proved that there exists in $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$ a $\mathfrak{sl}(3, \mathbb{R})$ -invariant non-degenerate bilinear form. This has as consequence the existence of an isomorphism $\varrho : \mathbb{R}^3 \oplus \mathbb{R}^{3*} \rightarrow (\mathbb{R}^3 \oplus \mathbb{R}^{3*})^*$ of $\mathfrak{sl}(3, \mathbb{R})$ -modules.

Then we have an isomorphism $\varrho(\mathbb{C}) : \mathbb{C}^3 \oplus \mathbb{C}^{3*} \rightarrow \mathbb{C}^3 \oplus \mathbb{C}^{3*}$ of $\mathfrak{sl}(3, \mathbb{C})$ -modules, that by Schur's Lemma, is just the multiple of the identity by a complex number when restricted to \mathbb{C}^3 and to another complex number when restricted to \mathbb{C}^{3*} . Furthermore, since $\varrho(\mathbb{C})$ is the complexification of ϱ we have that these numbers are real.

The result follows from the previous arguments and the fact that \mathbb{R}^3 and \mathbb{R}^{3*} belong to the nullcone of the inner product. \square

Lemma A.4. *Let $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ be inner products on $\mathfrak{sl}(3, \mathbb{R})$ and $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$, respectively. If we suppose that $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ are $\mathfrak{sl}(3, \mathbb{R})$ -invariant. Then, there exist $a_1, a_2 \in \mathbb{R}$ such that $a_1 \langle \cdot, \cdot \rangle_1 + a_2 \langle \cdot, \cdot \rangle_2$ is the Killing form of $\mathfrak{g}_{2(2)}$.*

Proof. Recall that Schur's Lemma implies that in \mathfrak{g} , a simple real Lie algebra with a simple complexification, any \mathfrak{g} -invariant non-degenerate symmetric bilinear form on \mathfrak{g} is unique up to a multiple, that is, the multiple by a scalar of the Killing form.

In particular, we have that $\langle \cdot, \cdot \rangle_1$ is a multiple of the Killing form of $\mathfrak{g}_{2(2)}(K)$ when restricted to $\mathfrak{sl}(3, \mathbb{R})$, this is $\langle X, Y \rangle_1 = c_1 K|_{\mathfrak{sl}(3, \mathbb{R})}(X, Y)$ for all $X, Y \in \mathfrak{sl}(3, \mathbb{R})$ and some non-zero $c_1 \in \mathbb{R}$.

On the other hand, from Lemma A.3 we have the existence of a non-zero scalar $c_2 \in \mathbb{R}$ such that $\langle \cdot, \cdot \rangle_2 = c_2 K|_{\mathbb{R}^3 \oplus \mathbb{R}^{3*}}(\cdot, \cdot)$.

Now, the result is a consequence of the previous arguments and Lemma A.3. \square

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